BULLETIN
DE
L'ACADÉMIE POLONAISE DES SCIENCES

SÉRIE DES SCIENCES
MATHÉMATIQUES, ASTRONOMIQUES ET PHYSIQUES

VOLUME VII
NUMERO 4

VARSOVIE 1959

PRINTED IN POLAND

PAŃSTWOWE WYDAWNICTWO NAUKOWE, WARSZAWA
MIODOWA 10

Nakład 1090 + 140 egz.	*Rękopis dostarczono 4. III. 1959*
Ark. wyd. 5 ark. druk. 5 $^{6}/_{16}$	*Podpisano do druku 3. VI. 1959*
Papier druk. sat. 80 g kl. III	*Druk ukończono w czerwcu 1959*
Format B5, 70×100 cm	*Zam. prod. 152/59 Cena zł 20.–*

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

MATHEMATICS

On the Asymptotic Behaviour of the Solutions of the Equation $\Delta u - \frac{\partial u}{\partial t} + c(x)u = 0$

by

A. SZYBIAK

Presented by T. WAŻEWSKI on February 2, 1959

We consider the equation

$$\Delta u - \frac{\partial u}{\partial t} + c(x)u = 0, \tag{1}$$

where

$$\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x^{i2}}, \quad x = \{x^1, \dots, x^m\}, \quad m \geqslant 1,$$

$c(x)$ is a bounded non-negative function which satisfies Hölder's condition in the m-dimensional Cartesian space $\mathcal{E}^m$. The fundamental solution of this equation is given by the formula

$$u(t, x, s, y) = \sum_{n=0}^{\infty} u_n(t, x, s, y), \tag{2}$$

where the u_n are represented by the formulae [1]:

$$\begin{aligned} u_0(t, x, s, y) &= \left(4\pi(t-s)\right)^{-\frac{m}{2}} \exp\left\{-\left(4(t-s)\right)^{-1}|x-y|^2\right\} \\ u_n(t, x, s, y) &= \int_s^t d\tau \int u_0(t, x, \tau, \xi) c(\xi) u_{n-1}(\tau, \xi, s, y) d\xi \quad (n = 1, 2, \dots). \end{aligned} \tag{3}$$

It may be proved by simple computation that the function

$$w(t, x, s, y) = \left\{2\pi a^{-1} \operatorname{sh}\left(2a(t-s)\right)\right\}^{-\frac{m}{2}}$$
$$\exp\left\{-\frac{a}{2} \operatorname{cth}\left(2a(t-s)\right)(|x|^2 + |y|^2) + x \cdot y \operatorname{sh}^{-1}\left(2a(t-s)\right) x \cdot y + b(t-s)\right\}$$

is the fundamental solution of the equation

$$\Delta u - \frac{\partial u}{\partial t} + (-a^2|x|^2 + b)u = 0 .$$

If $b > ma$, then we have

$$\lim_{t\to\infty} \exp\{-(b-ma)(t-s)\} w(t, x, s, y) = \exp\left\{-\frac{a}{2}(|x|^2+|y|^2)\right\}$$

and the inequality $-a^2|x|^2+b \leqslant b$ yields [2]:

$$w(t, x, s, y) < \exp\{b(t-s)\} u_0(t, x, s, y) . \tag{4}$$

We make use here of the fact that $\exp\{b(t-s)\}u_0(t, x, s, y)$ is the fundamental solution of the equation $\Delta u - \frac{\partial u}{\partial t} + bu = 0$, so that the inequality (4) follows by the formulae (2), (3).

We define on $\mathcal{E}^m \times [0, \infty)$ the function $C(x, R)$ which is equal to 1, if $|x| \leqslant R$, and to 0, if $|x| > R$. We put

$$v(t, x, s, y|\lambda, R) = \sum_{n=0}^{\infty} v_n(t, x, s, y|R) ,$$

where

$$v_0(t, x, s, y|R) = u_0(t, x, s, y) ,$$

$$v_n(t, x, s, y|R) = \int_s^t d\tau \int_{|x|<R} u_0(t, x, \tau, \xi) u_{n-1}(\tau, \xi, s, y|R) d\xi \quad (n = 1, 2, \ldots).$$

If a and b are chosen so that

$$\lambda > b, \quad R > \sqrt{b}/a , \tag{5}$$

then we have

$$\lambda C(x, R) > -a^2|x|^2 + b , \tag{6}$$

which implies

$$w(t, x, s, y) < v(t, x, s, y|\lambda, R) < \exp\{b(t-s)\} u_0(t, x, s, y) .$$

Summarizing the results we obtain

LEMMA 1. *If we may find for the given λ and R two numbers a and b such that they satisfy* (5) *and $b > ma$ then, by $t\to\infty$, $v(t, x, s, y|\lambda, R)$ tends to $+\infty$ exponentially.*

We shall prove

LEMMA 2. *For all $\lambda > 0$ and $R > 0$ we have $\lim_{t\to\infty} v(t, x, s, y|\lambda, R) = +\infty$ and the convergence is of the exponential order.*

Proof: λ being given we may choose R' such that there exists a and b for which

$$b > ma, \quad R' \geqslant \sqrt{b}/a, \quad \lambda' > b, \quad \lambda' = \lambda(R'/R)^2 \quad R'/R > 1 .$$

We deduce from (2) the formula

$$(7) \quad v_n(t, x, s, y|R') = \int_s^t d\tau_n \int_{|\xi_n|\leqslant R'} d\xi_n u_0(t, x, \tau_n, \xi_n) \int_s^{\tau_n} d\tau_{n-1} \int_{|\xi_{n-1}|\leqslant R'} d\xi_{n-1} \cdot$$
$$\cdot u_0(\tau_n, \xi_n, \tau_{n-1}, \xi_{n-1}) \ldots \int_s^{\tau_p} d\tau_{p-1} \int_{|\xi_{p-1}|\leqslant R'} d\xi_{p-1} u_0(\tau_p, \xi_p, \tau_{p-1}, \xi_{p-1}) \ldots$$
$$\ldots \int_s^{\tau_1} d\tau_1 \int_{|\xi_1|\leqslant R'} d\xi_1 u_0(\tau_2, \xi_2, \tau_1, \xi_1) u_0(\tau_1, \xi_1, s, y) .$$

We substitute

$$\xi_p = k\eta_p, \tau_p = k^2\vartheta_p , \quad \text{where} \quad k = R'|R(>1) .$$

By a simple computation we obtain

$$v_n(t, x, s, y|R') = k^{2n-m} v_n(k^{-2}t, k^{-1}x, k^{-2}s, k^{-1}y|R)$$

and hence

$$(8) \qquad v(t, x, s, y|\lambda', R') = v(k^{-2}t, k^{-1}x, k^{-2}s, k^{-1}y|\lambda, R)k^{-m} .$$

In view of (8) and Lemma 1 this yields our thesis.

THEOREM 1. *If $c(x) \geqslant 0$ in $\mathcal{E}^m$ and $c(x) \not\equiv 0$ then the corresponding fundamental solution of* (1) *tends to $+\infty$ by $t\to\infty$ and the convergence is of the exponential order and uniform in x and y on every compact.*

Proof: Without loss of generality we may assume that $c(0) > 0$. We choose λ and R in such a way that $c(x) \geqslant \lambda C(x, R)$ and we have

$$v(t, x, s, y|\lambda, R) \leqslant u(t, x, s, y) \leqslant \exp\{\max c(x)(t-s)\} u_0(t, x, s, y)$$

and the theorem follows by Lemma 2.

THEOREM 2. *If $u_s(t, x)$ is the solution of the Eq.* (1) *with the initial condition $u_s(s, x) = \varphi(x)$, where $\varphi(x) \geqslant 0$ and $\varphi(x) \not\equiv 0$ then $u_s(t, x)$ tends to $+\infty$ by $t\to\infty$ the convergence being of the exponential order and uniform on every compact.*

Proof: The theorem follows by the former and the formula

$$u_s(t, x) = \int_{\mathcal{E}^m} u(t, x, s, y)\varphi(y)dy .$$

The author expresses his sincere gratitude to Professor M. Krzyżański for his valuable advice.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] F. G. Dressel, *The fundamental solution of the parabolic equation, I and II*, Duke Math. Journ., **7** (1940), 186-203; ibid., **13** (1946), 61-70.

[2] M. Krzyżański et A. Szybiak, *Construction et l'étude de la solution fondamentale de l'équation linéaire parabolique dont le dernier coefficient est non borné*, Atti dell' Acad. Nat. dei Lincei (in press).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 4, 1959

MATHEMATICS

On the Embedding of One Space of Continuous Functions into Another

by

K. GĘBA and Z. SEMADENI

Presented by S. MAZUR on February 7, 1959

By $C(\Omega)$ we shall denote the Banach lattice of all real-valued continuous functions defined on a compact Hausdorff space Ω.

Ω and Ω_0 being compact Hausdorff spaces, we shall say that the space $C(\Omega_0)$ is B_+-contained in the space $C(\Omega)$ and we shall write $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$, if there exists a linear, one-to-one, norm-preserving and isotonical transformation T of $C(\Omega_0)$ onto a subspace X_0 of $C(\Omega)$.

Let e be the unit of $C(\Omega)$ (i. e. the function $e(t) = 1$ on Ω) and let e_0 be the unit of $C(\Omega_0)$. In general, T being isometrical and isotonical, we have

$$T(x \wedge y) \leqslant T(x) \wedge T(y) \leqslant T(x) \vee T(y) \leqslant T(x \vee y)$$

and

$$T(e_0) \leqslant e.$$

Thus, the relations

$$x \vee_0 y = T[T^{-1}(x) \vee T^{-1}(y)]$$

and

$$x \wedge_0 y = -[(-x) \vee_0 (-y)]$$

determine the relative supremum and infimum in X_0; similarly, the element Te_0 is the relative unit in X_0 *).

We shall denote by Ω_1 the set

$$\Omega_1 = \{t \in \Omega : (Te_0)(t) = 1\} \cap \bigcap_{x,y \in X_0} \{t \in \Omega : (x \vee_0 y)(t) = (x \vee y)(t)\}.$$

*) If X_0 is a closed linear subset of a Banach lattice X, it may be at the same time a lattice with respect to the order induced by X; however, the relative operations $\vee_0$ and $\wedge_0$ may differ from the operations $\vee$ and $\wedge$ induced by X.

Thus, Ω_1 is the set of all points of Ω for which the operations $\vee$ and $\vee_0$ and the units e and e_0 coincide, respectively. Obviously, Ω_1 is compact.

We shall say that $C(\Omega_0)$ is *MI*-contained in $C(\Omega)$ and we shall write $C(\Omega_0) \underset{\mathrm{MI}}{\subset} C(\Omega)$, if $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$, and if T and X_0 may be chosen so that

$$T(e_0) = e,$$

and

$$T(x \vee y) = Tx \vee Ty \quad \text{for all} \quad x, y \in C(\Omega_0).$$

In this case $\Omega_1 = \Omega$.

THEOREM 1. *Let* $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$, *let* T *be a linear, one-to-one, isometrical and isotonical transformation of* $C(\Omega_0)$ *onto a subspace* X_0 *of* $C(\Omega)$ *and let* Ω_1 *be defined as above. Then*

$$\|x\| = \sup\{|x(t)| : t \in \Omega_1\}$$

for every $x \in X_0$, *and there exists a continuous mapping of* Ω_1 *onto* Ω_0.

If Ω_0 and Ω are metrisable and if Ω_0 is a continuous image of a closed subset Ω_1 of Ω, then the converse theorem is also true, i. e. $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$ [2]; in general, $C(\Omega_0)$ may be isomorphic to no subspace of $C(\Omega)$, even if $\Omega_0 \underset{\mathrm{top}}{\subset} \Omega$ ([3], p. 527).

Considering the case $\Omega_1 = \Omega$ we obtain the following variant of a well known theorem ([6] p. 475, [5], [7] and [4], p. 421):

THEOREM 2. $C(\Omega_0) \underset{\mathrm{MI}}{\subset} C(\Omega)$ *if and only if there exists a continuous mapping of* Ω *onto* Ω_0.

THEOREM 3. *Let* $C(\Omega_0) \underset{B_+}{\subset} C(\Omega)$, *let* T *be as above, and let there exist a non-negative* *) *projection* P, *of norm* 1, *transforming* $C(\Omega)$ *onto* X_0.

Then $\Omega_0 \underset{\mathrm{top}}{\subset} \Omega$; *moreover, there exists a homeomorphism* φ *between* Ω_0 *and a closed subset* Ω_P *of* Ω *such that*

$$\big(T^{-1}Px\big)(\varphi^{-1}(t)) = x(t) \quad \textit{for} \quad t \in \Omega_P,$$

(which means that P *corresponds to the operation of restriction of a function* $x(t)$ *to* Ω_P; *in other words, there exists a simultaneous extension* **) *of the continuous functions on* Ω_P *to continuous functions on* Ω).

If $e \in X_0$, then the conditions $\|P\| = 1$ and $P \geqslant 0$ are equivalent.

THEOREM 4. Ω_0 *is homeomorphic to a retract of* Ω *if and only if* $C(\Omega_0) \underset{\mathrm{MI}}{\subset} C(\Omega)$ *and if there exists a non-negative projection of* $C(\Omega)$ *onto* X_0.

---

*) An operation P is termed non-negative if $P(x) \geqslant 0$ for every $x \geqslant 0$; P is termed a projection if $P^2 = P$.

**) In the sense of K. Borsuk [2] and S. Kakutani [5].

This theorem is a generalization of one due to K. Borsuk [1] and H. Yoshizawa [8].

The proofs and a detailed discussion will be published elsewhere [9], [10].

INSTITUTE OF MATHEMATICS, NICHOLAS COPERNICUS UNIVERSITY, TORUŃ
(INSTYTUT MATEMATYCZNY UNIWERSYTETU IM. M. KOPERNIKA, TORUŃ)
INSTITUTE OF MATHEMATICS, A. MICKIEWICZ UNIVERSITY, POZNAŃ
(INSTYTUT MATEMATYCZNY UNIWERSYTETU IM. A. MICKIEWICZA, POZNAŃ)

REFERENCES

[1] K. Borsuk, *Sur les retractes*, Fund. Math., **17** (1931), 152-170.

[2] — *Über Isomorphie der Funktionalräume*, Bull. Int. Acad. Pol. Sci. (1933), 1-10.

[3] M. M. Day, *Strict convexity and smoothness of normed spaces*, Trans. Amer. Math. Soc., **78** (1955), 516-528.

[4] Ky Fan, *Partially ordered additive groups of continuous functions*, Annals of Math., **51** (1950), 409-427.

[5] S. Kakutani, *Simultaneous extension of continuous functions considered as a positive linear operation*, Japan. J. Math., **17** (1940), 1-4.

[6] G. Silov, *Ideals and subrings of the ring of continuous functions* (in Russian), Doklady (A. N. URSS), **22** (1939), 7-10.

[7] M. H. Stone, *Applications of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., **41** (1937), 375-481.

[8] H. Yoshizawa, *On simultaneous extensions of continuous functions*, Proc. Imp. Acad. Tokyo, **20** (1944), 653-654.

[9] K. Gęba and Z. Semadeni, *On the M-subspaces of the Banach spaces of continuous functions*, Roczniki Uniwersytetu A. Mickiewicza, Poznań (in press).

[10] — *Spaces of continuous functions (V)*, Studia Math. (in press).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 4, 1959

MATHÉMATIQUE

Sur la notion du degré topologique pour une certaine classe de transformations multivalentes dans les espaces de Banach

par

A. GRANAS

Presénté par K. BORSUK le 7 Fevrier 1959

Notations. Désignons respectivement par E_a un espace quelconque fixé de Banach pour $a=\infty$ et son sous-espace n-dimensionnel pour $a=n<+\infty$. Soient $x_0\in E_a$, $A\subset E_a$ et ϱ un nombre positif; posons

$$V_a(x_0,\varrho)=\mathop{E}_{x\in E_a}(\|x-x_0\|<\varrho),\ V_a(A,\varrho)=\mathop{E}_{x\in E_a}\mathop{\exists}_{a\in A}(\|x-a\|<\varrho),$$

$$P_a=E_a\setminus\{0\},\ S_{a-1}(x_0,\varrho)=\begin{cases}\mathrm{Fr}\big(V_\infty(x_0,\varrho)\big) & \text{pour} \quad a=\infty\\ \mathrm{Fr}\big(V_n(x_0,\varrho)\big) & \text{pour} \quad a=n,\end{cases}$$

$$V_a^0=V_a(0,1),\qquad S_{a-1}^0=S_{a-1}(0,1).$$

Si E_n est un sous-espace de E_{n+1} contenant le centre de la sphère $S_n\subset E_{n+1}$, E_{n+1}^+, E_{n+1}^- désignent les demi-espaces fermés de E_{n+1} tels que $E_n=E_{n+1}^+\cap E_{n+1}^-$. Posons $S_n^+=S_n\cap E_{n+1}^+$, $S_n^-=S_n\cap E_{n+1}^-$.

Par minuscules latines $f,g,\dots,r,s,t$ nous désignons les transformations continues univalentes et par majuscules F,G — les transformations univalentes complètement continues.

Transformations multivalentes. Une transformation définie sur $A\subset E_a$ qui à chaque point $x\in A$ attache un ensemble non vide $\varphi(x)\subset E_a$ est dite *supérieurement semi-continue* si les conditions $x_n\to x$, $(x_n,x\in A)$, $y_n\to y$, $y_n\in\varphi(x_n)$ impliquent $y\in\varphi(x)$. Nous ne considérons dans cette note que les fonctions supérieurement semi-continues dont les valeurs sont des ensembles *convexes* dans E_a. Nous écrivons $\varphi\colon A\to E_a$ pour dire que φ est une transformation supérieurement semi-continue de A dans E_a; $\varphi\colon(A,B)\to(E_a,C)$ (où $B\subset A$, $C\subset E_a$) signifie de plus que $\varphi(B)\subset C$ où $\varphi(B)=\mathop{E}_{y\in E_a}\mathop{\exists}_{x\in B}\{y\in\varphi(x)\}$.

Nous dirons qu'une fonction multivalente $\Phi: A \to E_a$ est une *transformation multivalente complètement continue* *) (t. m. c. c.) lorsque l'image $\Phi(A')$ de chaque ensemble $A' \subset A$ borné dans E_a est relativement compact dans E_a.

Lemme 1. *Soit $A \subset E_\infty$ un ensemble borné et $\Phi: A \to E_\infty$ une t. m. c. c. de A. Pour chaque $\varepsilon > 0$ il existe un sous-espace $E_n \subset E_\infty$ et une transformation $\Phi_\varepsilon: A \to E_n$ tels que les inclusions:*

$$(1) \qquad \Phi_\varepsilon(x) \subset V_\infty(\Phi(x), \varepsilon), \quad \Phi(x) \subset V_\infty(\Phi_\varepsilon(x), \varepsilon).$$

ont lieu pour tous $x \in A$.

Champs vectoriels multivalents complètement continus dans E_a. Nous dirons qu'une transformation $\varphi: A \to C \subset E_a$ est un *champ vectoriel multivalent complètement continu* (c. v. m. c. c.) sur A dans E_a, (ou *déplacement multivalent complètement continu* (d. m. c. c.) de l'ensemble A dans E_a) si elle peut être representée sous la forme **)

$$(2) \qquad \varphi(x) = x - \Phi(x),$$

où $\Phi: A \to E_a$ est une t. m. c. c.

Deux c. v. m. c. c. $\varphi_1, \varphi_2: A \to C$ (A — borné, $C \subset E_a$, $\varphi_1(x) = x - \Phi_1(x)$, $\varphi_2(x) = x - \Phi_2(x)$) sont dits *homotopes*, $\varphi_1 \simeq \varphi_2$, lorsqu'il existe une famille des c. v. m. c. c. $\psi: A \times I \to C$ $\psi(x, t) = x - \Psi(x, t)$ (I — intervalle $[0, 1]$) telle, que pour chaque $x \in A$ on a $\Phi_1(x) = \Psi(x, 0)$, $\Phi_2(x) = \Psi(x, 1)$ et l'ensemble $\Psi(A, I)$ est relativement compact dans E_a.

Lemme 2. *Soit $\varphi: A \to E_a$ un d. m. c. c.; si A est un ensemble fermé et borné, $\varphi(A)$ est fermé.*

Notion de la caractèristique. Soit $S_{a-1} = S_{a-1}(x_0, \varrho)$, $V_a = V_a(x_0, \varrho)$ et $f, \varphi: S_{a-1} \to P_a$ champs complètement continus (respectivement unis et multivalents) sur S_{a-1}. Par $\gamma(f, S_{a-1})$ nous désignons la caractèristique [2] du champ f sur S_{a-1}. Un champ univalent $f: S_{a-1} \to P_a$ est dit *sélécteur* du champ multivalent $\varphi: S_{a-1} \to P_a$, (en symboles: $f \in \varphi$), si $f(x) \in \varphi(x)$ pour chaque $x \in S_{a-1}$.

Théorème 1a. *A chaque c. v. m. c. c. $\varphi: S_{a-1} \to P_a$ on peut associer un nombre entier $\gamma(\varphi, S_{a-1})$ d'une telle façon, que les conditions suivantes soient satisfaites*

I_a *si $f \in \varphi$, alors $\gamma(\varphi, S_{a-1}) = \gamma(f, S_{a-1})$;*

II_a *si les c. v. m. c. c. $\varphi_1, \varphi_2: S_{a-1} \to P_a$ sont homotopes $\varphi_1 \simeq \varphi_2$, alors $\gamma(\varphi_1, S_{a-1}) = \gamma(\varphi_2, S_{a-1})$. On appelle le nombre $\gamma(\varphi, S_{a-1})$ la caractèristique du c. v. m. c. c. φ sur S_{a-1}.*

---

*) En désignant par minuscules grecques φ, ψ, η les transformations multivalentes, nous réservons les majuscules Φ, Ψ pour les transformations multivalentes complètement continues.

**) $\varphi(x)$, $\psi(x)$ étant deux fonctions multivalentes, on désigne par $\eta(x) = t_1\varphi(x) + t_2\psi(x)$ (t_1, t_2 — nombres réels) la fonction multivalente qui à chaque point x fait correspondre l'ensemble des points de la forme $t_1 z_1 + t_2 z_2$ où $z_1 \in \varphi(x)$, $z_2 \in \psi(x)$; $\varphi(x)$, $\psi(x)$ étant convexes, $\eta(x)$ l'est aussi.

Démonstration. Les cas $\alpha = n$. Dans ce cas nous allons utiliser la théorie d'homologie de Vietoris pour les espaces métriques compacts avec les coefficients entiers. Si X, Y sont les espaces compacts et $f: X \to Y$, nous désignons par $f_*^{(i)}: H_i(X) \to H_i(Y)$ l'homomorphisme des groupes d'homologie, induit par f.

Soit $\varphi: S_{n-1} \to P_n$ une c. v. m. c. c. Désignons par $\Gamma = \Gamma_\varphi$ le graphique de la fonction φ et par $r: \Gamma \to S_{n-1}$, $s: \Gamma \to P_n$ ses projections sur S_{n-1} et $\varphi(S_{n-1})$ respectivement ($\Gamma_\varphi = \underset{z \in S_{n-1} \times P_n}{E} \{z = (x, y),\, y \in \varphi(x)\}$, $r(x, y) = x$, $s(x, y) = y$; $(x, y) \in \Gamma_\varphi$). L'ensemble Γ est compact dans $S_{n-1} \times P_n$ et la projection r verifie les conditions du Théorème de Vietoris [3] (toutes les images réciproques étant convexes) donc l'homomorphisme induit des groupes d'homologie $r_*^{(i)}: H_i(\Gamma) \to H_i(S_{n-1})$ est un isomorphisme sûr. Nous avons en particulier $r_* = r_*^{(n-1)}: H_{n-1}(\Gamma) \approx H_{n-1}(S_{n-1})$.

Posons $t(z) = \dfrac{s(z)}{\|s(z)\|}$ pour $z \in \Gamma$; nous avons $t: \Gamma \to S_{n-1}^0$

$$t_* = t_*^{(n-1)}: H_{n-1}(\Gamma) \to H_{n-1}(S_{n-1}^0), \qquad (r_*)^{-1}: H_{n-1}(S_{n-1}) \approx H_{n-1}(\Gamma)$$

$$\varphi_* = t_*(r_*)^{-1}: H_{n-1}(S_{n-1}) \to H_{n-1}(S_{n-1}^0).$$

Soient u et u^0 respectivement les générateurs des groupes $H_{n-1}(S_{n-1})$ et $H_{n-1}(S_{n-1}^0)$. Nous avons $\varphi_*(u) = \gamma \cdot u^0$ où γ est un nombre entier.

Nous definissons la caractéristique du c. v. m. c. c. $\varphi: S_{n-1} \to P_n$ en posant $\gamma(\varphi, S_{n-1}) = \gamma$. La démonstration des propriétés I_n et II_n ne diffère pas essentiellement de celle de Jaworowski [4].

Les cas $\alpha = \infty$. L'ensemble $\varphi(S_\infty)$ est fermé (voir lemme 2); désignons par ε un nombre plus petit que la demidistance d'ensemble $\varphi(S_\infty)$ du point 0. Selon le lemme 1 nous pouvons associer à ε un tel sous-espace $E_n \subset E_\infty$ et une telle transformation $\Phi_\varepsilon: S_\infty \to E_n$ que les inclusions (1) aient lieu pour chaque $x \in S_\infty$; nous pouvons supposer que $x_0 \in E_n$, donc $E_n \cap S_\infty = S_{n-1}$. Il en resulte qu'en posant $\varphi_\varepsilon(x) = x - \Phi_\varepsilon(x)$ pour $x \in S_{n-1}$ nous avons $\varphi_\varepsilon: S_{n-1} \to P_n$.

Nous définissons la caractèristique du champ $\varphi: S_\infty \to P_\infty$ en posant $\gamma(\varphi, S_\infty) = \gamma(\varphi_\varepsilon, S_{n-1})$. Le nombre $\gamma(\varphi, S_\infty)$ pour un E_n fixé ne dépend pas de la manière d'approximation de la fonction Φ par Φ_ε. En effet, soit Φ'_ε une autre fonction pour laquelle les inclusions (1) ont lieu et $\varphi'_\varepsilon(x) = x - \Phi'_\varepsilon(x)$. L'ensemble $V_\infty(\Phi(x), \varepsilon)$ est convexe, donc $x \bar{\in} \Psi(x, t) = t\Phi_\varepsilon(x) + (1-t)\Phi'_\varepsilon(x)$; en posant $\psi(x, t) = x - \Psi(x, t)$, $x \in S_{n-1}$ on obtient $\psi: S_{n-1} \times I \to P_n$, $\psi(x, 0) = \varphi_\varepsilon(x)$, $\psi(x, 1) = \varphi'_\varepsilon(x)$, $\varphi_\varepsilon \simeq \varphi'_\varepsilon$, $\gamma(\varphi_\varepsilon, S_{n-1}) = \gamma(\varphi'_\varepsilon, S_{n-1})$.

Le nombre $\gamma(\varphi, S_\infty)$ ne dépend pas du choix de E_n. Ceci résulte du lemme suivant:

LEMME 3. *Soit E_n un sous-espace de E_{n+1} contenant le centre d'une sphère $S_n \subset E_{n+1}$. Supposons que le c. v. m. c. c. $\varphi: S_n \to P_{n+1}$ satisfait à la*

condition suivante $\varphi(S_n^+) \subset E_{n+1}^+$, $\varphi(S_n^-) \subset E_{n+1}^-$ (*donc* $\varphi(S_{n-1}) \subset P_n$), $S_{n-1} = S_n^+ \cap S_n^-$ *et* $\varphi_0 = \varphi | S_{n-1}$. *Alors* $\gamma(\varphi, S_n) = \gamma(\varphi_0, S_{n-1})$.

Les propriétés I_∞ et II_∞ résultent de I_n et II_n respectivement.

Théorème 2_a. *Si le c. v. m. c. c.* $\varphi\colon (\overline{V}_a, S_{a-1}) \to (E_a, P_a)$ $\varphi(x) = x - \Phi(x)$ *vérifie la condition* $\gamma(\varphi_0, S_{a-1}) \neq 0$, *où* $\varphi_0 = \varphi | S_{a-1}$, *il existe un point* $x \in V_a$ *tel, que* $0 \in \varphi(x)$, *c'est à dire* $x \in \Phi(x)$.

Notion du degré topologique. Soit $\psi\colon \overline{V}_a \to E_a$ un d. m. c. c. et $y_0 \bar{\in} \psi(S_{a-1})$. Nous définissons alors le *degré topologique* $d(\psi, V_a, y_0)$ en posant $d(\psi, V_a, y_0) = \gamma(\varphi, S_{a-1})$ où $\gamma(\varphi, S_{a-1})$ est la caractéristique du c. v. m. c. c. $\varphi\colon S_{a-1} \to P_a$ défini par $\varphi(x) = \psi(x) - y_0$ pour $x \in S_{a-1}$.

Théorème $2'_a$. *Si* $\psi\colon \overline{V}_a \to E_a$ *est un d. m. c. c.*, $y_0 \bar{\in} \psi(S_{a-1})$ *et* $d(\psi, V_a, y_0) \neq 0$, *il en résulte que* $y_0 \in \operatorname{Int}(\psi(V_a))$.

UNIVERSITÉ NICOLAS COPERNIC, TORUŃ
(UNIWERSYTET MIKOŁAJA KOPERNIKA, TORUŃ)

OUVRAGES CITÉS

[1] J. Leray et J. Schauder, *Topologie et équations fonctionelles*, Ann. de l'Ecole Norm. Sup., **51** (1934), 45-78.

[2] E. Rothe, *Theorie der Ordnung und der Vektorfelder in Banachschen Räumen*, Comp. Math., **5** (1937), 177-197.

[3] E. G. Begle, *The Vietoris mapping theorem for bicompact spaces*, Ann. of Math., **51** (1950), 534-543 (voir Theorème 1).

[4] J. W. Jaworowski, *Some consequences of the Vietoris mapping theorem*, Fund. Math., **45** (1958), p. 261-272.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No, 4, 1959

MATHEMATICS

Remarks on Linear Metric Montel Spaces

by

S. ROLEWICZ

Presented by S. MAZUR on February 7, 1959

This note contains some theorems concerning the isomorphic structure of M type B_0-spaces (linear metric locally convex complete spaces) *).

We recall that a B_0-space X is called of M type (Montel space) if every linear operation U mapping space X into an arbitrary Banach space Y is completely continuous (see [1]).

The B_0-space X is said to be a nuclear space (of N-type) if every linear operation U of the space X into an arbitrary B-space Y is absolutely continuous **).

The B_0-space is said to be of S-type (Schwartz space) if for every linear operation U of the space X into an arbitrary Banach space Y there exist a Banach space Y_0 and a linear operation V_1 of the space X into Y_0 and a linear operation V_2 mapping $V_1(X)$ into Y such that $U = V_2 V_1$ and V_2 is completely continuous.

The definition and basic properties of the spaces of types M, S, N are given in [2]-[4], [10].

The S-type spaces may also be characterized in another way, by the approximate dimension introduced by Kolmogorov in [5] or by Pełczyński's method [9].

In [5] Kolmogorov considers for every linear metric space X the class $\Phi(X)$ of all functions $\varphi(\varepsilon)$ defined for $\varepsilon > 0$ and such that for every compact $K \subset X$ and for an arbitrary neighbourhood U of zero there is such ε_0 that for each $\varepsilon < \varepsilon_0$ there exist $N \leqslant \varphi(\varepsilon)$ of points $x_1, \ldots, x_N$ in X

---

*) A definition and basic properties of B_0-spaces are given in [7].

**) We call the linear operation $U(x)$ mapping F-space X into B-space Y absolutely continuous, if for every unconditional convergent series $\sum_{n=1}^{\infty} x_n$, where $x_n \in X$, the series $\sum_{n=1}^{\infty} \|U(x_n)\|$ is convergent ([3]).

such that $K \bigcup_{1 \leqslant m \leqslant N} (x_m + \varepsilon U)$. The spaces X and Y have the same approximative dimension $d_a(X) = d_a(Y)$ if $\Phi(X) = \Phi(Y)$.

Let X be B_0-space, and let the topology in X be given by a sequence of pseudonorms $\|x\|_1, \|x\|_2, \ldots$ such that $\|x\|_i \leqslant \|x\|_{i+1}$ for $x \in X$ and $i = 1, 2, \ldots$

We introduce, as in [9], a function $a_{i,j}(X, \varepsilon)$ for $\varepsilon > 0$ $(i, j = 1, 2, \ldots)$ in the following way:

$a_{i,j}(X, \varepsilon) = n$ — where n is the maximal number of the points x_r $(r = 1, 2, \ldots, n)$, such that for every $r, p = 1, 2, \ldots, n$ and $r \neq p$ $\|x_r - x_p\|_i \geqslant \varepsilon$ and $\|x_r\|_{i+j} < 1$

or

$a_{i,j}(X, \varepsilon) = \infty$ — if there exists no finite maximum number n.

When $\varepsilon \to 0$, the approximate dimension and the asymptotic property of the matrix $a_{i,j}(X, \varepsilon)$ are an isomorphic invariant but this invariant may be applied only for the isomorphic classification of S-type spaces. This is a consequence of the following

THEOREM 1 *). *The B_0-space X is of S-type if and only if the class $\Phi(X)$ is not empty and if and only if for each $i = 1, 2, \ldots$ for almost all j the functions $a_{i,j}(X, \varepsilon)$ for positive ε are finite.*

Moreover, the class $\Phi(X)$ consists of all functions $\varphi(\varepsilon)$ such that for every $\delta > 0$ and $i = 1, 2, \ldots$ for almost all j

$$\lim_{\varepsilon \to 0} \frac{a_{i,j}(X, \delta\varepsilon)}{\varphi(\varepsilon)} = 0 .$$

It may easily be proved [2], [3] that each nuclear space is a Schwartz space and each Schwartz space is a Montel space. The inverse implication does not hold in the general case. However, we have

THEOREM 2. *Every infinite dimensional M-type B_0-space contains an infinite dimensional nuclear subspace with an unconditional basis.*

Let $a_{m,n}$ be a double sequence of non-negative numbers. By $M(a_{m,n})$ we denote the space of all real sequences $x = \{t_n\}$ such that

$$\|x\|_m = \sup_n a_{m,n} |t_n| < +\infty \quad (m = 1, 2, \ldots)$$

with topology given by pseudonorms $\|x\|_m$ **).

THEOREM 3. *Let X be a nuclear B_0-space with an unconditional basis ***) e_n and let the topology in X be given by the sequence of pseudo-*

*) The proof of this theorem was communicated to me by C. Bessaga and A. Pełczyński.

**) Similar spaces $K(a_{m,n})$ defined by the matrices $a_{m,n}$ have eben considered by G. Köthe. But in $K(a_{m,n})$ pseudonorms are defined in the following way: $||x||_m = \sum_{n=1}^{\infty} a_{m,n} |t|_n$.

***) The basis e_n of the space X is called unconditional, if the convergence of series $\sum_{n=1}^{\infty} t_n e_n$ implies the convergence of series $\sum_{n=1}^{\infty} t_{p_n} e_{p_n}$, where p_n is an arbitrary permutation of natural numbers.

norms $|x|_m$ $(m = 1, 2, \ldots)$. *Then the space* X *is isomorphic to the space* $M(\|e_n\|_m)$.

It may be that, of all linear metric Schwartz spaces, the nuclear spaces may be characterized by "sufficiently large" classes $\Phi(X)$. But now we know only the following theorems:

THEOREM 4. *Let* X *be a nuclear* B_0*-space with unconditional basis. If* $\lim\limits_{\varepsilon\to 0} \dfrac{e^{\lambda/\varepsilon}}{\varphi(\varepsilon)} = 0$ *for some* $\lambda > 0$, *then* $\varphi \in \Phi(X)$.

THEOREM 5. *Let* X *be a* B_0*-space with unconditional basis* e_n, *let the topology in* X *be given by the sequence of pseudonorms* x_m *and let the sequences* $\dfrac{\|e_n\|_m}{\|e_n\|_{m+1}}$ *be monote tending to* 0. *If there exists* $\varphi(\varepsilon) \in \Phi(X)$ *such that* $\lim\limits_{\varepsilon\to 0} \varphi(\varepsilon) e^{-\frac{1}{(\varepsilon)^k}} = 0$ *for some* k, *then* X *is a nuclear space.*

The proofs of the theorems given in this note will be published in Studia Mathematica.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] S. Banach, *Theorie des opérations linéaires*, Varsovie, 1932.

[2] J. Dieudonné et L. Schwarth, *La dualité dans les espaces* ($\mathcal{F}$) et ($\mathcal{DF}$), Ann. Inst. Fourier (Grenoble), **1** (1950), 61-101.

[3] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc., **16** (1955).

[4] — *Sur les espaces* ($\mathcal{F}$) *et* ($\mathcal{DF}$), Summa Brasil. Math., **3** (1954), 57-121.

[5] A. N. Kolmogorov, *On the linear dimension of topological vectorial spaces* (in Russian), DAN USSR, **120** (1958), 239-241.

[6] G. Köthe, *Die Stufenräume, eine einfache Klasse linear volkommener Räume*, Math. Zeitr., **51** (1949), 317-345.

[7] S. Mazur et W. Orlicz, *Sur les espaces métriques linéaires*, Stud. Math., **10** (1948), 184-208.

[8] — ibid., **13** (1953), 137-179.

[9] A. Pełczyński, *On the approximation of S-spaces by finite dimensional spaces*, Bull. Acad. Polon. des Sci., Cl. III, **5** (1957), 879-881.

[10] D. A. Raikov, *On a certain property of nuclear spaces* (in Russian), Usp. Math. Nauk, **12** (1957), 231-236.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No, 4, 1959

MATHEMATICS

General Method of Limitation of the Borel Type

by

L. WŁODARSKI

Presented by S. MAZUR on February 7, 1959

In [1] and [2], I have dealt with Borel's B_a methods of limitation. We term the expression

$$B_a(t, x) = ae^{-t} \sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} \xi_n$$

as the transform of the sequence $x = \{\xi_n\}$ with regard to the B_a method, while the generalized limit ξ of the sequence $x = \{\xi_n\}$ by the B_a method is defined by the formula: $\lim\limits_{t\to\infty} B_a(t, x) = \xi$. In papers [1] and [2] I was concerned with these methods only for a of the character 2^k $(k = 0, \pm 1, \pm 2, ...)$. These are new theorems with an optional positive a. B_a methods are continuous, permanent and capable of displacement to the right. If the sequence x is limited by one of the B_a methods to the number ξ, while its transform exists with regard to the Abelian method, then sequence x is limited by the Abelian method to the number ξ. Thus the B_a methods are in agreement with the Abelian method. If sequence x is limited by the B_a method to the number ξ, and its transformate exists with regard to the method $B_{\theta a}$ $(0 < \theta < 1)$, then sequence x is limited by the $B_{\theta a}$ method to the number ξ. The proof of the theorem in question is based on the regularity of the transformation $F = W_\theta\{f\}$, defined by the following formula

$$F(t) = W_\theta\{f\} = \theta e^{-t} \int_0^{\infty} e^{v} f(v) L^{-1}\{s^{\theta-1} e^{-vs^\theta}\} dv\,,$$

where L^{-1} stands for a transformation which is the converse of a Laplace transformation. It can easily be shown on the basis of the properties of the Laplace transformation that, particularly with $\theta = 1/2$, we obtain

the transformation $W: F(t) = \frac{e^{-t}}{2\sqrt{\pi t}} \int_0^\infty \exp\left(-\frac{v^2}{4t}+v\right) f(v)\,dv$ (see [2], p. 150) used in the proof of the agreement of the B_a method taking indices of the character $a = 2^k$.

The proof of the regularity of the transformation W_θ, i. e. of the property on the basis of which from $\lim_{v\to\infty} f(v) = l$ we obtain the result $\lim_{t\to\infty} F(t) = l$, is based on the use of the Efros formula [3], from which it results, in particular, that

$$L^{-1}\{s^{\theta-1}F(s^\theta)\} = \int_\theta^\infty f(v) L^{-1}\{s^{\theta-1}e^{-vs^\theta}\}\,dv,$$

where

$$0 < \theta < 1,\ F(s) = \int_\theta^\infty e^{-vs} f(v)\,dv,\ \int_\theta^\infty e^{-vs}|f(v)|\,dv < \infty \quad \text{for} \quad R(s) \geqslant \gamma > 0.$$

As may be seen, the B_a methods are in agreement amongst themselves with respect to all a, while as continuous methods they have a series of regular properties (see [4]).

The results presented here were communicated at the meeting of the Łódź section of the Polish Mathematical Society on October 10th, 1957. The proofs will be published in Annales Polonici Mathematici.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] L. Włodarski, *Propriété des méthodes continues de limitation du type de Borel*, Bull. Acad. Polon. Sci., Cl. III, **4** (1956), 173-175.

[2] — *Sur les méthodes continues du type de Borel*, Ann. Pol. Math., **4** (1958), 137-164.

[3] — *Sur une formule de Efros*, Studia Math., **13** (1953), 183-187.

[4] — *Sur les méthodes continues de limitation*, Studia Math., **14** (1954), 161-199.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 4, 1959

MATHEMATICS

On a Certain Class of Topological Division Algebras

by
W. ŻELAZKO

Presented by S. MAZUR on February 7, 1959

Introduction. S. Mazur announced in [7] two results concerning real normed algebras. The first asserts that only a real normed division algebra is the field of reals, the complex field, or the division algebra of real quaternions. The first published proof of this theorem (in the case of complex scalars) was given by I. M. Gelfand [5], whose proof was based upon the abstract theory of analytical functions. By a similar method A. Arens [1] proved this theorem for locally convex division algebras. In the proofs of Gelfand and Arens the existence of linear functionals in the algebras considered is essential. In 1952 S. Kametani [6] and L. Tornheim [10] gave elementary proofs of the Mazur theorem. The fact that there exists a homogeneous norm in the algebras considered is essential for these proofs. In this paper some theorems of the Mazur type are set forth. Neither the existence of linear functionals, nor that of a homogeneous norm is assumed here. We merely assume that the division algebra has the p-homogeneous norm, or by applying a certain theorem of A. Pełczyński (see below), we get Mazur's theorem under the assumption of the existence of a submultiplicative norm, or of the bounded neighbourhood of 0. We also get the generalization of the theorems of R. E. Edwards [4], S. Mazur ([7], theorem 2), and R. Arens [1]; the latter in the case of m-convex division algebras.

Definitions and notations. The linear-metric space X is called normed space if the metric function $\varrho(x, y) = \|x-y\|$, where the non-negative function $\|x\|$ called the norm satisfies the following conditions ([3], pp. 35, and 37):

(a) $\|x\| = 0$ if and only if $x = 0$,

(b) $\|x\| = \|-x\|$ for every $x \in X$,

(c) $\|x+y\| \leqslant \|x\| + \|y\|$ for every $x, y \in X$,

(d) if (a_n) is a sequence of scalars tending to 0, then, for every $x \in X$, $\lim_n \|a_n x\| = 0$,

(e) if $\lim_n \|x_n\| = 0$, $x_n \in X$, $n = 1, 2, \ldots$, then $\lim_n \|a x_n\| = 0$ for every scalar a.

The non-negative function $\|x\|$ satisfying conditions (b)-(e) is called the pseudonorm.

The norm (pseudonorm) $\|x\|$ is called p-homogeneous if it satisfies $\|ax\| = |a|^p \|x\|$ for every $x \in X$, and every scalar a (we assume as scalars either complex, or real numbers). It is easy to see that it must be $0 < p \leqslant 1$. If $p = 1$, then the norm is called homogeneous.

The linear metric space X is called locally bounded if there exists the bounded neighbourhood V of 0, i. e. such open set V containing 0, that for every sequence of scalars (a_n) tending to 0 and for every sequence (x_n), $x_n \in V$, the sequence $(a_n x_n)$ tends to zero. For every complete locally bounded space there exists an equivalent p-homogeneous norm, i. e. such norm $\|x\|$ that $\|x - x_0\| \to 0$ if and only if $\varrho(x_n, x_0) \to 0$ (see [9]).

The algebra R is called locally bounded if it is locally bounded as the linear space. For locally bounded algebras the following holds:

THEOREM. *Let R be a complete, metric algebra over the real or complex scalars. Let R possess the unit element e; then the following statements are equivalent*:

(a) *R is locally bounded.*

(b) *The metric ϱ in R is submultiplicative, i. e. $\varrho(xy, 0) \leqslant \varrho(x, 0)\varrho(y, 0)$ holds for every $x, y \in R$.*

(c) *In R an equivalent submultiplicative p-homogeneous norm exists.*

This theorem (unpublished) is due to A. Pełczyński.

The m-convex algebra is a topological algebra in which topology is introduced by the family $\|x\|_t$, $t \in T$ of submultiplicative pseudonorms (for similar definition see [8]).

In the sequel we assume R to be the topological algebra over the real scalars with the unit e.

Results. THEOREM 1. *If R is a complete metric division algebra for which* (a), (b), *or* (c) *of Pełczyński's theorem hold, then R is either the field of reals, or the complex field, or the division algebra of real quaternions.*

THEOREM 2. (Generalization of Edwards theorem [4]). *If R is complete in the submultiplicative norm $\|x\|$, for which $\|x^{-1}\| = \|x\|^{-1}$ holds for every x invertible in R, then R is either the field of reals, or the complex field, or the division algebra of real quaternions.*

THEOREM 3. (Generalization of Mazur's theorem [7]), (see also [2]).

If R is complete in the norm $\|x\|$, so that $\|xy\| = \|x\|\|y\|$ for every x, y, R, then R is the field of reals, the complex field, or the division algebra of real quaternions.

THEOREM 4. (Generalization of the Arens theorem [1]). *If R is an m-convex division algebra, then it is either the field of reals, or the complex field, or the division algebra of the real quaternions.*

The proofs and the developement of the theory will appear in Studia Mathematica.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] R. Arens, *Linear topological division algebras*, Bull. Amer. Math. Soc., **53** (1947), 623-630.

[2] S. Aurora, *Multiplicative norms for metric rings*, Pacific J. Math., **7** (1957), 1279-1323.

[3] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[4] R. E. Edwards, *Multiplicative norms on Banach algebras*, Proc. Cambridge Phil. Soc., **47** (1951), 473-474.

[5] I. M. Gelfand, *Normierte Ringe*, Rec. Math., **9** (1941), 3-24.

[6] S. Kametani, *An elementary proof of the fundamental theorem of normed fields*, J. Math. Soc. Japan, **4** (1952), 96-99.

[7] S. Mazur, *Sur les anneaux linéaires*, C. R. Paris, **207** (1938), 1025-1027.

[8] E. Michael, *Locally multiplicatively-convex topological algebras*, Mem. Amer. Math. Soc. **11** (1952), 1-79.

[9] S. Rolewicz, *On a certain class of linear metric spaces*, Bull. Acad. Polon. Sci., Cl. III, **5** (1957), 479-484.

[10] L. Tornheim, *Normed fields over the real and complex fields*, Michigan Math. J. **1** (1952), 61-68.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 4, 1959

MATHÉMATIQUE

Problème aux limites aux dérivées tangentielles pour l'équation elliptique

par

W. POGORZELSKI

Présenté par T. WAŻEWSKI le 11 Février 1959

1. Introduction. Le problème aux limites aux dérivées tangentielles, posé pour la première fois par H. Poincaré [1], consiste en la recherche d'une fonction harmonique à l'intérieur d'un domaine plan qui, en tout point de la courbe limitant ce domaine, vérifie une relation linéaire entre les valeurs limites: 1° de la dérivée dans la direction de la normale; 2° de la dérivée dans la direction de la tangente; 3° de la fonction elle-même.

Le problème de Poincaré, qui a fait l'objet des recherches de plusieurs auteurs, a été approfondi par B. Hvédélidzé [2] et I. Vécoua [3]. L'auteur du présent travail a généralisé le problème de Poincaré dans le cas d'une relation non-linéaire entre les valeurs limites des dérivées au bord du domaine donné (voir [4]).

Dans le présent travail nous poserons et résoudrons le problème aux limites aux dérivées tangentielles pour une équation elliptique du second ordre de la forme générale

$$(1)\qquad \hat{\Psi}(u) = \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha(A)\frac{\partial u}{\partial x_\alpha} + c(A)u$$
$$= F\left(A, u, \frac{\partial u}{\partial x_1}, \dots \frac{\partial u}{\partial x_n}\right)$$

dans l'espace à n dimensions.

Dans nos études nous nous appuierons principalement sur les propriétés suivantes des derivées tangentielles du potentiel de simple couche, relatif à l'équation (1), démontrées dans notre travail [5].

PROPRIÉTÉ 1. Si la densité $\varphi(Q)$, définie sur la surface fermée S de Liapounoff, vérifie une condition de Hölder

$$|\varphi(Q)-\varphi(Q_1)| < k_\varphi r_{QQ_1}^{h_\varphi}$$

($0 < h_\varphi < 1$, $k_\varphi > 0$), alors la dérivée du potentiel de simple couche, présentée par l'intégrale de surface

$$V(A) = \iint\limits_{S(Q)} \Gamma(A, Q)\varphi(Q)\,dQ$$

(Γ désigne la solution fondamentale de l'équation (1)) dans la direction d'une tangente s_P au point P de la surface S, tend uniformément vers une limite

$$\lim_{A\to P} V_{s_P}(A) = V_{s_P}(P) = \iint\limits_{S(Q)} \Gamma_{s_P}(P, Q)\varphi(Q)\,dQ\,,$$

où l'intégrale singulière a le sens de la valeur principale de Cauchy.

Propriété 2. Sous les mêmes hypothèses, la fonction limite de la dérivée tangentielle vérifie une condition de Hölder

$$|V_{s_P}(P) - V_{s_{P_1}}(P_1)| < (C_1 \sup|\varphi| + C_2 k_\varphi) r_{PP_1}^{h'_\varphi}\,,$$

où $h'_\varphi = \min(h_\varphi, \theta h_s, h, \varkappa)$, h est l'exposant de Hölder concernant les coefficients de l'équation (1), $\varkappa$ est un exposant de Liapounoff [5], h_s est un exposant concernant l'angle entre les tangentes aux points P et P_1 de la surface S, θ est une constante positive arbitraire, inférieure à l'unité, C_1 et C_2 sont des constantes positives, indépendantes de la fonction φ.

Les raisonnements que nous présenterons sont analogiques à ceux que nous avons développé dans notre travail [6] consacré au problème des dérivées tangentielles pour l'équation parabolique normale dans l'espace à $n+1$ dimensions. C'est pourquoi nous présenterons les résultats de nos recherches d'une façon abrégée.

2. Énoncé du problème. Soient q champs des tangentes

$$\{s_P^{(1)}\}, \{s_P^{(2)}\}, \ldots \{s_P^{(q)}\} \qquad (1 \leqslant q \leqslant n-1) \tag{2}$$

définies sur la surface fermée S limitant le domaine borné Ω dans l'espace euclidien à n dimensions. Nous posons le problème de la détermination d'une fonction $u(A)$ qui:

1° vérifie l'équation (1) en tout point intérieur $A(x_1, \ldots x_n)$ du domaine Ω;

2° satisfait à une condition limite aux dérivées tangentielles de la forme

$$\frac{du}{dT_P} + g(P)u(P) = G[P, u(P), u_{s_P^{(1)}}(P), \ldots u_{s_P^{(q)}}(P)] \tag{3}$$

en tout point $P \in S$; du/dT_P désigne la valeur limite de la dérivée transversale au point P et $u_{s_P^{(\alpha)}}(P)$ — la valeur limite de la dérivée dans la direction tangentielle $s_P^{(\alpha)}$.

On admet les hypothèses suivantes:

I. Les coefficients $a_{\alpha\beta}(A), b_\alpha(A), c(A)$ sont définis dans la région fermée $\Omega+S$, où ils vérifient les conditions de Hölder

$$|a_{\alpha\beta}(A)-a_{\alpha\beta}(A_1)|<\text{const.}\, r_{AA_1}^h$$

(4) $$|b_\alpha(A)-b_\alpha(A_1)|<\text{const.}\, r_{AA_1}^h;\quad |c(A)-c(A_1)|<\text{const.}\, r_{AA_1}^h,$$

r_{AA_1} désignant la distance euclidienne des points A, A_1.

II. La forme quadratique $\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A) X_\alpha X_\beta$ est définie-positive dans $\Omega+S$.

III. La surface S vérifie les conditions de Liapounoff, dont l'une concernant l'angle $\varDelta$ entre les normales aux deux points arbitraires P et Q, a la forme

(5) $$\varDelta(P,Q)<\text{const.}\, r_{PQ}^{\varkappa}\qquad (0<\varkappa\leqslant 1).$$

IV. La fonction $F(A, u_0, u_1, \dots u_n)$ est définie dans la région fermée

(6) $$A\in\Omega+S;\ |u_\alpha|\leqslant R\qquad (\alpha=0,1,2,\dots n)$$

où elle vérifie la condition de Hölder par rapport à toutes les variables.

V. La fonction $G(P, u_0, u_1, \dots u_q)$ est définie dans la région fermée

(7) $$P\in S;\quad |u_\alpha|\leqslant R;\quad (\alpha=0,1,\dots q)$$

où elle vérifie la condition de Hölder-Lipschitz de la forme

(8) $$|G(P, u_0, u_1, \dots u_n)-G(P', u_0', u_1', \dots u_n')|\leqslant k_G\Big[r_{PP'}^{h_G}+|u_0-u_0'|^{h_G'}+\sum_{\alpha=1}^{n}|u_\alpha-u_\alpha'|\Big].$$

VI. La fonction $g(P)$ est définie sur la surface S, où elle vérifie la condition de Hölder

(9) $$|g(P)-g(P')|\leqslant k_G r_{PP'}^{h_G}$$

et, en outre, est telle, que le problème homogène aux limites $dv/dT_P+g(P)\,v=0$ pour l'équation homogène linéaire $\hat{\Psi}(v)=0$ n'admet qu'une solution nulle $v=0$.

VII. Tout champ des tangentes donné $\{s_P^{(\alpha)}\}$ vérifie la condition suivante concernant l'angle entre deux directions du champ aux deux points arbitraires P et Q de la surface S:

(10) $$(s_P^{(\alpha)}, s_Q^{(\alpha)})<\text{const.}\, r_{PQ}^{h_s}\qquad (0<h_s\leqslant 1).$$

3. Solution du problème. Nous allons chercher la solution du problème sous la forme d'une somme

$$(11)\qquad u(A) = -\iiint\limits_{\Omega} \Gamma(A, B)\lambda_n^{-1}(B)F\left[B, u(B), \frac{\partial u}{\partial \xi_1}, \dots \frac{\partial u}{\partial \xi_n}\right]dB + \iint\limits_{S} \Gamma(A, Q)\varphi(Q)dQ$$

du potentiel de charge spatiale et du potentiel de simple couche de densité inconnue $\varphi(Q)$ [7]. On a posé

$$(12)\qquad \lambda_n(B) = 2(n-2)(\sqrt{\pi})^n\left[\Gamma\left(\frac{n}{2}\right)\sqrt{\det|a^{\alpha\beta}(B)|}\right]^{-1}.$$

$\Gamma(A, B)$ désigne la solution fondamentale de l'équation $\hat{\Psi}(u) = 0$ [7]. En demandant que la fonction (11) vérifie la condition limite (3), sous l'hypothèse que la densité φ vérifie une condition de Hölder, on obtient, d'après les travaux [5] et [7], l'équation suivante

$$(13)\qquad -\frac{1}{2}\lambda_n(P)\varphi(P) + \iint\limits_{S}\left[\frac{d\Gamma(P, Q)}{dT_P} + g(P)\Gamma(P, Q)\right]\varphi(Q)dQ$$
$$-\iiint\limits_{\Omega}\left\{\frac{d}{dT_P}[\Gamma(P, B)] + g(P)\Gamma(P, B)\right\}\lambda_n^{-1}(B)F\left[B, u(B), \frac{\partial u}{\partial \xi_1}, \dots \frac{\partial u}{\partial \xi_n}\right]dB$$
$$= G[P, u(P), u_{s_P^{(1)}}(P), \dots u_{s_P^{(q)}}(P)],$$

où

$$(14)\qquad u_{s_P^{(\alpha)}}(P) = -\iiint\limits_{\Omega}\Gamma_{s_P^{(\alpha)}}(P, B)\lambda_n^{-1}(B)F\left[B, u(B), \frac{\partial u}{\partial \xi_1}, \dots \frac{\partial u}{\partial \xi_n}\right]dB + \iint\limits_{S}\Gamma_{s_P^{(\alpha)}}(P, Q)\varphi(Q)dQ; \qquad (\alpha = 1, 2, \dots q).$$

Notre problème est donc ramené à la solution du système d'équations intégro-différentielles (11) et (13) aux deux fonctions inconnues: $u(A)$ dans la région Ω et $\varphi(P)$ sur la surface S. Les équations précédentes contiennent les fonctions aux singularités fortes sous le signe des intégrales de surface dans les expressions (14).

Pour résoudre le système d'équations (11) et (13), considérons le système suivant d'équations intégrales

$$(15)\qquad u_0(A) = -\iiint\limits_{S}\Gamma(A, B)\lambda_n^{-1}(B)F[B, u_0(B), \dots u_n(B)]dB + \iint\limits_{S}\Gamma(A, Q)\varphi(Q)dQ$$

$$
\begin{aligned}
u_\nu(A) = & -\iiint_\Omega \Gamma_{x_\nu}(A, B)\lambda_n^{-1}(B)F[B, u_0(B), \dots u_n(B)]\,dB \\
& + \iint_S \Gamma_{x_\nu}(A, Q)\varphi(Q)\,dQ; \quad (\nu = 1, 2, \dots n) \\
& -\frac{1}{2}\lambda_n(P)\varphi(P) + \iint_S \left[\frac{d\Gamma(P, Q)}{dT_P} + g(P)\Gamma(P, Q)\right]\varphi(Q)\,dQ \\
= & \iiint_\Omega \left\{\frac{d}{dT_P}[\Gamma(P, B)] + g(P)\Gamma(P, B)\right\}\lambda_n^{-1}(B)F[B, u_0(B), \dots u_n(B)]\,dB \\
& + G[P, u_0(P), \overline{u}_{s_P^{(1)}}(P), \dots \overline{u}_{s_P^{(q)}}(P)],
\end{aligned}
$$

aux $n+2$ fonctions inconnues $u_0(A), \dots u_n(A), \varphi(P)$. On a posé

$$
\begin{aligned}
(16) \qquad \overline{u}_{s_P^{(\alpha)}}(P) = & -\iiint_\Omega \Gamma_{s_P^{(\alpha)}}(P, B)\lambda_n^{-1}(B)F[B, u_0(B), \dots u_n(B)]\,dB \\
& + \iint_S \Gamma_{s_P^{(\alpha)}}(P, Q)\varphi(Q)\,dQ
\end{aligned}
$$

$(\alpha = 1, 2, \dots q)$. Le système d'équations intégrales (15) aux fortes singularités étant irrésoluble par les méthodes d'analyse classique, nous démontrerons l'existence de la solution par l'application du théorème topologique de J. Schauder (voir [4] ou [6]).

Soit donc Λ un espace fonctionnel composé de tous les systèmes $[u_0(A), \dots u_n(A), \varphi(P)]$ de fonctions continues réelles, définies, soit dans la région fermée $\Omega+S$, soit dans S. On définit, comme d'habitude, la somme des deux points et le produit d'un point par le nombre réel à l'aide des formules: $[u_0, \dots u_n, \varphi]+[u_0', \dots u_n', \varphi'] = [u_0+u_0', \dots u_n+u_n', \varphi+\varphi']$, $\gamma[u_0, \dots u_n, \varphi] = [\gamma u_0, \dots \gamma u_n, \gamma\varphi]$, la norme d'un point $U = [u_0, \dots u_n, \varphi]$ par la formule: $\|U\| = \sum_{\nu=0}^{n} \sup|u_\nu(A)| + \sup|\varphi(P)|$, et la distance des deux points par la formule: $\delta(U, V) = \|U-V\|$.

L'espace Λ est donc espace de Banach. Considérons maintenant dans l'espace Λ un ensemble E de tous les points $U[u_0(A), \dots u_n(A), \varphi(P)]$ vérifiant les conditions suivantes:

$$
(17) \qquad |u_\nu(A)| \leqslant R; \quad |\varphi(P)| \leqslant \varrho; \quad |\varphi(P)-\varphi(P_1)| \leqslant k_\varphi r_{PP_1}^{h_\varphi} \quad (\nu = 0, 1, \dots n)
$$

R étant un nombre positif donné, figurant dans les inégalités admises (6), ϱ et k_φ étant deux constantes positives arbitraires, non fixées pour le moment; l'exposant h_φ est une constante positive fixée arbitrairement, mais sous la condition que les inégalités suivantes soient satisfaites:

$$
(18) \qquad 0 < h_\varphi \leqslant \min(h_G, h_g, h, \varkappa); \quad 0 < h_\varphi < \min(h_s, h_G').
$$

L'ensemble E est évidemment fermé et convexe.

En tenant compte de la forme des équations intégrales (15), transformons l'ensemble E par les relations suivantes

(19) $$v_\nu(A) = \hat{X}_\nu[u_0, u_1, \dots u_n, \varphi]; \qquad (\nu = 0, 1, 2, \dots n)$$

(19′) $$-\frac{1}{2}\lambda_n(P)\psi(P) + \iint\limits_S \left[\frac{d\Gamma(P, Q)}{dT_P} + g(P)\Gamma(P, Q)\right]\psi(Q)\,dQ$$

$$= \iiint\limits_\Omega \left\{\frac{d}{dT_P}[\Gamma(P, B)] + g(P)\Gamma(P, B)\right\}\lambda_n^{-1}(B)F[B, u_0(B), \dots u_n(B)]\,dB$$

$$+G[P, v_0(P), \overline{u}_{s_P}^{(1)}(P), \dots \overline{u}_{s_P}^{(q)}(P)]$$

où $\hat{X}_\nu[u_0, \dots u_n, \varphi]$ désignant les membres droits des $n+1$ équations premières successives (15). En nous appuyant sur les propriétés des potentiels relativs à l'équation (1), démontrées dans les travaux [5], [7] et sur les inégalités admises (8), (9), (18), nous concluons que les fonctions $v_\nu(A)$ et $\psi(P)$ verifient les inégalités

$$|v_0(A)| < c_1 M_F + c_2 \sup|\psi|; \qquad |v_\nu(A)| < c_1' M_F + c_2' \sup|\varphi| + c_3' k_\varphi$$

(20) $$|\psi(P)| < K_1 M_F + K_2 M_G; \qquad |\psi(P) - \psi(P_1)| <$$

$$< \{D_1 \sup|\psi| + D_2 k_G[a_1 + a_2(M_F + \varrho)^{h_G'} + a_3 M_F + a_4\varrho + a_5 k_\varphi]\} r_{PP_1}^{h_\varphi}$$

où

$$M_F = \sup|F|, \qquad M_G = \sup|G|;$$

$$c_1, c_2, c_1', c_2', c_3', K_1, K_2, D_1, D_2, a_1, a_2, a_3, a_4, a_5$$

sont des constantes positives, indépendantes des fonctions F, G, u_ν, φ. Il en résulte que l'ensemble E' transformé de l'ensemble E par les relations (19) fera partie de cet ensemble si les constantes du problème vérifient les inégalités suivantes

(21) $$c_1 M_F + c_2\varrho \leqslant R; \qquad c_1' M_F + c_2'\varrho + c_3' k_\varphi \leqslant R; \qquad K_1 M_F + K_2 M_G \leqslant \varrho;$$
$$D_1\varrho + D_2 k_G[a_1 + a_2(M_F + \varrho)^{h_G'} + a_3 M_F + a_4\varrho + a_5 k_\varphi] \leqslant k_\varphi.$$

Le choix des constantes positives ϱ et k_φ étant arbitraire, nous voyons que les conditions (21) seront toujours satisfaites, si les constantes M_F, M_G, k_G, caractérisant les fonctions données F, G, sont suffisamment petites.

L'ensemble transformé E' est évidemment compact, en outre, on peut démontrer d'une façon analogue à celle donnée dans le travail [5], que la transformation définie par les relations (19) et (19′) est continue dans l'espace Λ. Les conditions (21) étant satisfaites, nous en concluons, d'après le theorème de Schauder, l'existence d'au moins un point $U[u_0^*, u_1^*, \dots u_n^*, \varphi^*]$, qui est invariant relativement à la transformation (19) et (19′). Ce système de fonctions $[u_0^*(A), \dots u_n^*(A), \varphi^*(P)]$ présente

précisément une solution du système d'équations intégrales (15). Nous en concluons que la fonction trouvée $u_0^*(A)$ est une solution du problème aux limites proposés, c. à d. qu'elle vérifie l'équation (1) en tout point intérieur $A \in \Omega$, et satisfait à la condition limite (3) en tout point $P \in S$.

4. Cas particulier du domaine plan. Les considérations précédentes s'appliquent aussi au cas du domaine plan ($n = 2$), sous la condition de substituer pour la quasi-solution [5]

$$w^M(A, B) = \log\Big[\sum_{\alpha,\beta=1}^{n} a^{\alpha\beta}(M)(x_\alpha - \xi_\alpha)(x_\beta - \xi_\beta)\Big]$$

et en n'introduisant qu'un champ de tangentes à la courbe C, qui limite le domaine D. On a alors $h_s = \varkappa$.

Il est intéressant de faire une étude plus approfondie du cas de l'équation homogène

$$\sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(A)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\beta=1}^{n} b_\alpha(A)\frac{\partial u}{\partial x_\alpha} + c(A)u = 0 \tag{22}$$

et de la condition limite linéaire

$$\frac{du}{dT_P} + g(P)u + G(P)\frac{du}{d_{sP}} = f(P) \tag{23}$$

en tout point $P \in C$. On admet pour les coefficients $a_{\alpha\beta}(A)$, $b_\alpha(A)$, $c(A)$ les conditions (4), pour les fonctions $g(P)$, $G(P)$, $f(P)$ aussi les conditions de Hölder et pour la courbe C la condition (5). En cherchant la solution du problème de Poincaré (23) sous la forme d'un potentiel relatif à l'équation (22)

$$u(A) = \int_C \Gamma(A, Q)\varphi(Q)\,dQ \tag{24}$$

on arrive à l'équation intégrale linéaire

$$\frac{2\pi}{\sqrt{\det|a^{\alpha\beta}(A)|}}\varphi(P) + \int_C \left[\frac{d\Gamma(P,Q)}{dT_P} + g(P)\Gamma(P,Q) + G(P)\Gamma_s(P,Q)\right]\varphi(Q)\,dQ = f(P) \tag{25}$$

à la singularité forte figurant dans l'expression $\Gamma_s(P, Q)$, l'intégrale ayant le sens d'une valeur principale de Cauchy. L'équation (25) a la forme

$$\varphi(s) = f_1(s) + \int_C N(s,\sigma)\operatorname{cotg}\frac{\pi}{L}(s-\sigma)\varphi(\sigma)\,d\sigma \tag{26}$$

où s et σ sont les coordonnées curvilignes des points P et Q sur la courbe C, L désigne sa longueur et $N(s,\sigma)$ est une fonction (de période L) donnée sur C, vérifiant la condition de Hölder. On sait bien que l'équation de la forme réelle (26) peut être amenée à l'équation de la forme complexe

$$\mu(t) = \mathcal{F}(t) + \int_K \frac{F(t,\tau)}{\tau - t}\mu(\tau)\,d\tau \tag{27}$$

à la fonction inconnue $\mu(t)$, t et τ étant des variables complexes correspondant aux points sur la circonférence K de l'intégration, $F(t,\tau)$ est une fonction complexe vérifiant une condition de Hölder. L'équation (27) a été étudiée d'une façon très approfondie [8] vu qu'elle est liée au problème de Hilbert; nous en déduirons une étude approfondie du problème (23), notamment les conditions d'existence de la solution du problème proposé.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

OUVRAGES CITÉS

[1] H. Poincaré, *Mécanique celeste*, vol. III, Paris, 1910.

[2] B. Hvédelidzé, *Sur le problème aux limites de Poincaré* (en russe), C. R. Acad. SSSR, **30** (1941), 195-198.

[3] I. N. Vécoua, *Problèmes aux limites dans la théorie des équations elliptiques* (en russe), C. R. Acád., SSSR, 1940.

[4] W. Pogorzelski, *Problème aux limites de Poincaré généralisé*, Ann. Polon. Mathem., **2** (1955) 257-270.

[5] — *Propriétés des derivées tangentielles d'une intégrale de l'équation elliptique*, Ann. Polon. Mathem. (sous presse).

[6] — *Problème aux limites aux derivés tangentielles pour l'équation parabolique*, Ann. Scient. de l'Ecole Normale Sup. Paris, **75** (1958), 19-35.

[7] — *Étude de la solution fondamentale de l'équation elliptique*, Ann. Polon. Mathem., **3** (1957), 247-284.

[8] N. I. Muskhelishvili, *Singular integral equations*, P. Nordhoff N. V., Gröningen, Holland, 1953.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys. — Vol. VII, No. 4, 1959

MATHEMATICS

Absolute Calculus of Segments and Its Metamathematical Implications

by

W. SZMIELEW

Presented by K. BORSUK on February 12, 1959

Introduction. We regard as *elementary* geometry that part of geometry which can be formalized within elementary logic (first-order predicate calculus). We shall be concerned here with formalized systems: $\mathcal{E}_n$ of elementary n-dimensional *Euclidean* geometry, $\mathcal{H}_n$ of elementary n-dimensional *hyperbolic (Bolyai-Lobachevskian)* geometry, and $\mathcal{A}_n$ of elementary n-dimensional *absolute* geometry which is understood to be the common part of $\mathcal{E}_n$ and $\mathcal{H}_n$. A detailed description of $\mathcal{E}_n$ for $n = 2$ is found in [6]; for $n > 2$ the dimension axioms, A11 and A12, should be appropriately modified. The systems $\mathcal{H}_n$ and $\mathcal{A}_n$ have the same symbolism as $\mathcal{E}_n$ and differ from $\mathcal{E}_n$ only in that Euclid's axiom, A8, has been replaced by its negation or omitted, respectively. The main result stated by Tarski in [6] is the representation theorem for $\mathcal{E}_n$ (see theorem 1); the proof of this theorem is based upon the elementary *Euclidean calculus of segments*, the essential point of which is Euclid's theory of proportion (see [1], pp. 51 ff and 241 ff). The main result in [4] is the representation theorem for $\mathcal{H}_n$ (theorem 2.15), the proof of which is based upon the elementary *hyperbolic calculus of segments* due to the author (see [4], pp. 32 ff) *).

The aim of this paper is to construct in $\mathcal{A}_n$ a new geometrical algorithm, the *absolute calculus of segments*, which will provide a convenient apparatus for the proofs of the representation theorems for both Euclidean and hyperbolic geometries. This algorithm seems to present some interest independent of the implication to the representation problem. In particular, in the Euclidean case, it coincides with the Euclidean calculus and thus nearly the whole Euclidean theory of proportion

*) The other hyperbolic algorithm, Hilbert's *end calculus*, refers not to segments but to pencils of parallel axes (see [1], pp. 159 ff).

belongs to absolute geometry; more accurately, the existence of the fourth proportional to any three given segments is shown to be the only fundamental property of proportion which cannot be proved without applying Euclid's axiom.

Every model $\mathfrak{M} = \langle \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D} \rangle$ of $\mathcal{A}_n$ (and thus of $\mathcal{E}_n$ and $\mathcal{H}_n$ as well) is formed by an arbitrary non-empty set $\boldsymbol{A}$, a ternary relation $\boldsymbol{B}$ (the *betweenness relation*), and a quaternary relation $\boldsymbol{D}$ (the *equidistance relation*) among elements *(points)* of $\boldsymbol{A}$. In sections **1**-**7** we construct the absolute calculus of segments; the whole discussion in these sections refers to an arbitrary model $\mathfrak{M}$ of $\mathcal{A}_n$ for $n \geqslant 2$. In sec. **8** we consider an arbitrary model of $\mathcal{E}_n$ and discuss the relation between the absolute and Euclidean algorithms. In sec. **9** the same is done for the hyperbolic case. We end, in sec. **10**, with the representation theorem for $\mathcal{A}_n$.

1. Class $\boldsymbol{S}$ of free segments. By a *segment* we understand any non ordered couple pq of two distinct points p, q in $\boldsymbol{A}$. Two segments pq and $p'q'$ are *congruent* if and only if $\boldsymbol{D}(pqp'q')$. The set of all segments congruent to a given segment pq is called the *free segment* determined by pq and is denoted by $[pq]$. Free segments will be represented by variables $X, Y, Z, \ldots$ and the set of all free segments will be denoted by $\boldsymbol{S}$.

2. Relation $<$. For arbitrary free segments X and Y we assume

(I) $X < Y$ if and only if $\boldsymbol{B}(pqr)$, $[pq] = X$, and $[pr] = Y$ for some distinct points p, q, r in $\boldsymbol{A}$.

LEMMA 1. *$\langle \boldsymbol{S}, < \rangle$ is a non-empty simply ordered system without the last element.*

3. Auxiliary operation $\oplus$. Given two free segments X and Y, consider the free segment Z constructed in the following way: for some Lambert quadrilateral $pqrs$, in which $\sphericalangle p$, $\sphericalangle q$, and $\sphericalangle r$ are three right angles (Fig. 1), let $X = [pq]$, $Y = [qr]$, and $Z = [qs]$. Clearly, the segment Z thus defined does not always exist (since the Lambert quadrilateral $pqrs$ cannot always be constructed). If, however, Z exists, it is uniquely determined by X and Y (independently of the choice of $pqrs$) and we then put $X \oplus Y = Z$. To express the fact that $X \oplus Y$ does or does not exist, we shall write $X \oplus Y \in \boldsymbol{S}$, $X \oplus Y \notin \boldsymbol{S}$, respectively.

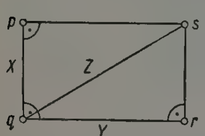

Fig. 1

LEMMA 2. *The operation $\oplus$ satisfies the following conditions:*

(i) *if $X, Y, X \oplus Y \in \boldsymbol{S}$, then $X \oplus Y = Y \oplus X$;*

(ii) *if $X, Y, Z, X \oplus Y, (X \oplus Y) \oplus Z \in \boldsymbol{S}$, then $Y \oplus Z \in \boldsymbol{S}$ and $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$;*

(iii) *if $X, Z \in \boldsymbol{S}$, then $X < Z$ if and only if $X \oplus Y = Z$ for some $Y \in \boldsymbol{S}$.*

The proofs of (i) and (iii) are obvious. In the proof of (ii) a three-dimensional construction is involved (see [4], proof of Lemma 2.1 (iii)). We know also a two-dimensional proof of (ii), but in the latter proof the two cases, Euclidean and hyperbolic, are separately considered, which is fully correct nevertheless not quite satisfactory.

4. Auxiliary operation Q. By an *angle* MN we understand the non-ordered pair of half-lines M and N which are supposed to have a common origin (the *vertex* of the angle MN) and to be non-collinear. Given any four free segments X, Y, X', Y' we put $\boldsymbol{Q}(XYX'Y')$ if and only if either $X = Y$ and $X' = Y'$ or else, for some angle MN with a vertex p and for some points x, x' on M and y, y' on N, we have $X = [px]$, $Y = [py]$, $X' = [px']$, $Y' = [py']$, while $\sphericalangle pxy$ and $\sphericalangle px'y'$ are two right angles (Fig. 2).

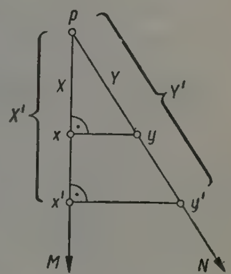

Fig. 2

LEMMA 3. *The relation $\boldsymbol{Q}$ satisfies the following conditions*:

(i) *if $\boldsymbol{Q}(XYX'Y')$ and $X \leqslant X'$, then $\boldsymbol{Q}(XX'YY')$*;
(ii) *if $\boldsymbol{Q}(XYX'Y')$ and $X \oplus X'$, $Y \oplus Y' \in \boldsymbol{S}$, then $\boldsymbol{Q}\big(XY(X \oplus X')(Y \oplus Y')\big)$.*

The proof of the lemma is based upon some simple property of the half-rotations introduced by Hjelmslev (see § 10 in [2]) and seems to be the most essential point of the whole construction.

5. Relation P (of proportion). Given a point p and any straight lines K and L, we put $\boldsymbol{O}(pKL)$ if and only if the perpendicular through p to K coincides with the perpendicular through p to L (Fig. 3).

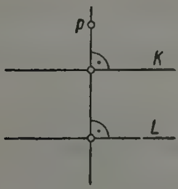

Fig. 3

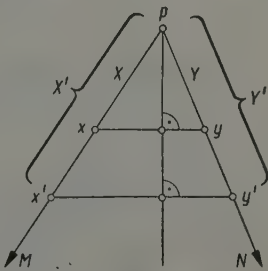

Fig. 4

Let MN be an arbitrary angle with a vertex p. Given any four free segments X, Y, X', Y', we take points x, x' on M and y, y' on N (Fig. 4) in such a way that $X = [px]$, $Y = [py]$, $X' = [px']$, $Y' = [py']$. We define:

$$\boldsymbol{P}_{MN}(XYX'Y') \text{ if and only if } \boldsymbol{O}\big(p\boldsymbol{L}(xy)\boldsymbol{L}(x'y')\big),$$

provided $\boldsymbol{L}(rs)$, for $r \neq s$, is the straight line determined by points r and s.

LEMMA 4. *For any arbitrary angle MN we have*

$$\boldsymbol{P}_{MN}(XYX'Y') \text{ if and only if } \boldsymbol{Q}(XYX'Y') \text{ or } \boldsymbol{Q}(YXY'X').$$

The proof is essentially based upon Lemma 3(i).

Thus, the relation $\boldsymbol{P}_{MN}$ does not depend on the angle MN, and we can put

$$\boldsymbol{P}(XYX'Y') \text{ if and only if } \boldsymbol{P}_{MN}(XYX'Y').$$

The relation $\boldsymbol{P}$ will be referred to as the *proportion*. To concentrate our attention we can assume that MN is a fixed, e. g. right, angle.

LEMMA 5. *The relation $\boldsymbol{P}$ satisfies the following conditions*:

(i) $\boldsymbol{P}(XYXY)$;
(ii) *if* $\boldsymbol{P}(XYX'Y')$ *and* $\boldsymbol{P}(XYX''Y'')$, *then* $\boldsymbol{P}(X'Y'X''Y'')$;
(iii) *if* $\boldsymbol{P}(XYX'Y')$, *then* $\boldsymbol{P}(XX'YY')$;
(iv) *if* $\boldsymbol{P}(XYX'Y')$ *and* $X\oplus X', Y\oplus Y' \in \boldsymbol{S}$, *then* $\boldsymbol{P}(XY(X\oplus X')(Y\oplus Y'))$;
(v) *if* $\boldsymbol{P}(XYX'Y')$ *and* $X < X'$, *then* $Y < Y'$;
(vi) *if* $\boldsymbol{P}(XYX'Y')$ *and* $X'' < X'$, *then* $\boldsymbol{P}(XYX''Y'')$ *for some* $Y'' \in \boldsymbol{S}$.

The conditions (i)-(vi) with the exception of (iii) and (iv) result immediately from the definition of $\boldsymbol{P}$. By applying Lemma 4 we reduce the conditions (iii) and (iv) to the conditions (i) and (ii) of Lemma 3.

The properties (i)-(v) of the relation $\boldsymbol{P}$ correspond to the well known fundamental properties of Euclidean proportion. The property (vi) replaces the stronger Euclidean property of the existence of the fourth proportional, the latter not being derivable in $\mathcal{H}_n$. It is worth while to notice that the properties (i)-(iii), (v) and (vi) are sufficient to derive the property

(1) if $\boldsymbol{P}(XYY'X')$ and $\boldsymbol{P}(YZZ'Y')$, then $\boldsymbol{P}(XZZ'X')$,

the latter corresponding, by (iii), to a special case of the Pascal theorem (theorem 10 in [1]) *).

With Lemma 5 the geometrical part of our construction is over. The subsequent discussion is of purely algebraic character.

To justify the definitions in Section **6** we mention three further properties of $\boldsymbol{P}$ easily derivable from Lemmas 1, 2, and 5, the second by applying an elementary continuity axiom.

(2) If $\boldsymbol{P}(XYX'Y')$ and $\boldsymbol{P}(XYX'Y'')$, then $Y' = Y''$.

(3) For every $X, Z \in \boldsymbol{S}$ we have $\boldsymbol{P}(XYYZ)$ for some $Y \in \boldsymbol{S}$.

(4) If $\boldsymbol{P}(XYYZ)$ and $\boldsymbol{P}(XYY'Z')$, then $Y = Y'$.

---

*) Confront the Euclidean proof of the property (1) in [1], p. 243, and the absolute proof of the Pascal theorem, e. g. in [3].

6. Operations $+$ and $\cdot$. We fix an arbitrary free segment U which henceforth will be referred to as the *unit* segment. Now, we put

(II) $$X \cdot Y = Z \quad \text{if and only if} \quad \boldsymbol{P}(UXYZ).$$

The *multiplication* $\cdot$ is clearly not always performable.

By (3) and (4), for a given free segment X there is a unique free segment Y such that $Y \cdot Y = X$; we put $Y = \sqrt{X}$. Now, the operation $+$ is defined by the following stipulation:

(III) $$X + Y = Z \quad \text{if and only if} \quad \sqrt{X} \oplus \sqrt{Y} = \sqrt{Z}.$$

The *addition* $+$ again is not always performable. On the other hand, we easily derive from Lemma 5 that it is independent from the choice of the unit segment U.

As usual, we put $X - Y = Z$, whenever $X = Y + Z$, and $X : Y = Z$, whenever $X = Y \cdot Z$.

Let us remark that $\oplus$ itself could serve as the operation of addition since the fundamental theorem on the absolute calculus (see Section **7**) applies to $\oplus$ as well as to $+$. In Sections **8** and **9**, however, the latter operation will prove to be more adequate for our discussion. It is worth while to notice that Lemma 5 (iv) remains valid if "$\oplus$" is replaced by "$+$".

Lack of space prevents us to list here all the fundamental properties of the operations $+$ and $\cdot$, such as commutativity, associativity, distributivity, etc.; they readily follow from Lemmas 1, 2 and 5. Only some existence properties of $:$ are mentioned to clear the theorem of sec. **7**.

(5) $U : X \in \boldsymbol{S}$ for some $X < U$.
(6) If $U : X \in \boldsymbol{S}$ and $Y > X$, then $U : Y \in \boldsymbol{S}$.
(7) If $U : X \in \boldsymbol{S}$, then $U : Y \in \boldsymbol{S}$ for some $Y < X$.

7. Fundamental theorem on absolute calculus of segments. As a consequence of Lemmas 1, 2, 5, the definitions (I), (II), (III), and the elementary continuity axioms we obtain

Theorem 1. *For an arbitrary model $\mathfrak{M}$ of $\mathcal{A}_n$ $(n \geqslant 2)$ the system $\mathfrak{S} = \langle \boldsymbol{S}, +, \cdot, < \rangle$ can be imbedded in a real closed field $\bar{\mathfrak{S}} = \langle \bar{\boldsymbol{S}} +, \cdot, < \rangle$ in such a way that $\underset{X \in \boldsymbol{S}}{E}(X < U) = \underset{X \in \bar{\boldsymbol{S}}}{E}(0 < X < U)$ (where 0 is the zero element of the field $\bar{\boldsymbol{S}}$). In fact, (α) $\bar{\mathfrak{S}}$ is up to isomorphism uniquely determined by $\mathfrak{S}$; (β) U is the unit element of the field $\bar{\mathfrak{S}}$, (γ) if $U : X \in \boldsymbol{S}$ for every $X \in \boldsymbol{S}$ (Euclidean case), then $\boldsymbol{S}$ consists of all positive elements in $\bar{\boldsymbol{S}}$; (δ) if $U : X \notin S$ for some $X \in \boldsymbol{S}$ (hyperbolic case), then $\boldsymbol{S} = \underset{X \in \bar{\boldsymbol{S}}}{E}(0 < X < R)$ where $R > U$ and $U : R$ is the last element X in $\boldsymbol{S}$ satisfying the condition $U : X \notin \boldsymbol{S}$.*

The proof of this theorem is easy, though lengthy and laborious. The continuity axioms are used to prove that the ordered field $\bar{\mathfrak{S}}$ is real closed, and to derive from (5)-(7) the existence of $U:R$.

8. Operations $+$ and $\cdot$ in the Euclidean case. Let us assume now that $\mathfrak{M}$ is a model of $\mathcal{E}_n$. Then $\boldsymbol{O}(pKL)$ if and only if $K\|L$; therefore, the relation $\boldsymbol{P}$ coincides with the Euclidean proportion and the operation $\cdot$ coincides with the Euclidean multiplication ${}^{(\mathrm{E})}\cdot$. Let us denote the Euclidean addition by ${}^{(\mathrm{E})}+$; thus

$X^{(\mathrm{E})}+Y=Z$ if and only if $\boldsymbol{B}(pqr), [pq]=X, [qr]=Y,$ and $[pr]=Z$ for some points p, q, r in $\boldsymbol{A}$.

By applying the Pythagorean theorem we readily obtain

$X+Y=Z$ if and only if $\sqrt{X}\oplus\sqrt{Y}=\sqrt{Z}$ if and only if $X^{(\mathrm{E})}+Y=Z$.

In conclusion, the absolute system $\langle \boldsymbol{S}, +, \cdot, < \rangle$ coincides with the Euclidean system $\langle \boldsymbol{S}, {}^{(\mathrm{E})}+, {}^{(\mathrm{E})}\cdot, < \rangle$.

9. Operations $+$ and $\cdot$ in hyperbolic case. Let us assume, in turn, that $\mathfrak{M}$ is a model of $\mathcal{H}_n$. The hyperbolic multiplication ${}^{(\mathrm{h})}\cdot$ can be de-

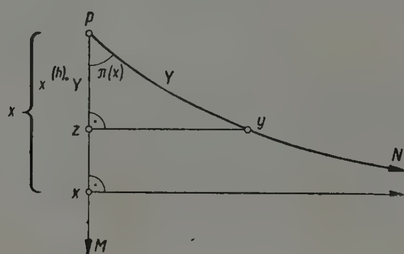

Fig. 5

fined as follows (see [4], lemma 2.4): Let Π be the Lobachevskian function. Given two free segments X and Y, we consider an angle MN with a vertex p such that $\Pi(X)=[MN]$ (Fig. 5) as well as points x on M and y on N such that $X=[px]$ and $Y=[py]$, and we put $X\,{}^{(\mathrm{h})}\!\cdot Y=[pz]$ provided z is the perpendicular projection of y upon M. By $[MN]$ we understand here the free angle consisting of all angles congruent to MN, and Π is treated as a function correlating free angles with free segments. The hyperbolic addition ${}^{(\mathrm{h})}+$ is given by the formula

$$X\,{}^{(\mathrm{h})}+Y=Z \text{ if and only if } {}^{(\mathrm{h})}\sqrt{X}\oplus{}^{(\mathrm{h})}\sqrt{Y}={}^{(\mathrm{h})}\sqrt{Z}.$$

By applying Lemma 4 and the properties of the operation ${}^{(\mathrm{h})}\cdot$ we derive the formula

(8) $$X\cdot Y=Z \text{ if and only if } X\,{}^{(\mathrm{h})}\!\cdot Y=Z\,{}^{(\mathrm{h})}\!\cdot U,$$

Note of the Editor: In Fig. 5 the upper three small x's should be capital letters.

which readily implies

$$X+Y=Z \quad \text{if and only if} \quad X \overset{(h)}{+} Y = Z\,.$$

In conclusion, the function f

$$f(X) = X \overset{(h)}{\cdot} U \quad \text{for} \quad X \in S\,,$$

isomorphically maps the hyperbolic system $\langle S, \overset{(h)}{+}, \overset{(h)}{\cdot}, < \rangle$ onto the absolute system $\langle \underset{X \in S}{E} (X < U), +, \cdot, < \rangle$ *).

The absolute operation $\cdot$ depends on the selected unit segment U:

$$\cdot = \cdot_U\,.$$

Since, for every E is S, we can find a D in S such that

$$U > D \quad \text{implies} \quad Z - Z \overset{(h)}{\cdot} U < E$$

(see formula (8)), the hyperbolic multiplication $\overset{(h)}{\cdot}$ proves to be the limit operation for the class of the operation $\cdot_U$ as the unit segment U increases without bound.

10. Representation theorem for $\mathcal{H}_n$.

THEOREM 2. $\mathfrak{M}$ *is a model of* $\mathcal{H}_n$ $(n \geqslant 2)$ *if and only if it is isomorphic either with the Cartesian space* $\mathfrak{C}_n(\mathfrak{F})$ *(Euclidean case) or with the Klein space* $\mathfrak{K}_n(\mathfrak{F})$ *(hyperbolic case) over some real closed field* $\mathfrak{F}$.

In one direction the proof is based upon the fact that every two real closed fields are elementarily equivalent (see [5]), in the other direction — upon the fundamental theorem on absolute calculus of segments. For the definitions of $\mathfrak{C}_n(\mathfrak{F})$ and $\mathfrak{K}_n(\mathfrak{F})$ see [6] and [4], respectively.

The result stated in Theorem 2 is known (theorem 3.1 in [4]) but the other proof is based upon two separately constructed geometrical algorithms.

When analyzing the proof of Theorem 2 we see that the rectangular co-ordinates introduced on the base of the absolute calculus of segments lead in the Euclidean case to the Cartesian model and in the hyperbolic case to the Klein model (based upon a circle with radius R greater than one). Thus, from the point of view of the absolute calculus the Klein model is the hyperbolic analogue of the Cartesian model. Moreover, in

*) We recall (see [4], pp. 33 ff) that in any model of full hyperbolic geometry (with the non-elementary axiom of continuity) we have

$$X \overset{(h)}{+} Y = Z \quad \text{if and only if} \quad \cos|\Pi(X)| + \cos|\Pi(Y)| = \cos|\Pi(Z)|\,,$$
$$X \overset{(h)}{\cdot} Y = Z \quad \text{if and only if} \quad \cos|\Pi(X)| \cdot \cos|\Pi(Y)| = \cos|\Pi(Z)|\,,$$

provided $|[MN]|$ is the measure of the free angle $[MN]$, for arbitrary MN, and $+$ and $\cdot$ on the right side are the ordinary operations on real numbers.

both cases, Euclidean and hyperbolic, the primitive formulas $\boldsymbol{B}(pqr)$ and $\boldsymbol{D}(pqrs)$ are algebraically expressed in terms of the co-ordinates of the points involved.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] D. Hilbert, *Grundlagen der Geometrie*, 8th ed. Stuttgart, 1956.

[2] J. Hjelmslev, *Einleitung in die allgemeine Kongruenzlehre I*, Copenhagen, 1929.

[3] — *Neue Begründung der ebenen Geometrie*, Math. Annalen, **64** (1907), 449-474.

[4] W. Szmielew, *Some metamathematical problems concerning elementary hyperbolic geometry*. The Axiomatic Method, Proceedings of an International Symposium, University of California, Berkeley, December 1957, pp. 30-52, Amsterdam (in press).

[5] A. Tarski, *A decision method for elementary algebra and geometry*. 2nd ed., Berkeley and Los Angeles, 1951.

[6] — *What is elementary geometry?* The Axiomatic Method, Proceedings of an International, Symposium, University of California, Berkeley, December 1957, Amsterdam (in press).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

MATHEMATICS

On Necessary and Sufficient Conditions for the Validity of the Strong Law of Large Numbers Expressed in Terms of Moments

by

M. FISZ

Presented by E. MARCZEWSKI on February 16, 1959

1. Introductory remarks. Let $\{X_i\}$ $(i = 1, 2, ...)$ be a sequence of random variables.

DEFINITION. We say that the sequence $\{X_i\}$ $(i = 1, 2, ...)$ obeys the strong law of large numbers (SLLN) if there exists such a sequence of constants $\{c_n\}$ that the relation

$$P\left(\lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=1}^{n} X_i - c_n\right) = 0\right) = 1$$

holds.

The problem of finding necessary and sufficient conditions for the validity of the SLLN for sequences of independent random variables has not yet found a final and satisfactory solution. A full account of the present state of research in this field is given in the expository paper of Chung [2], Loeve's monography [4] and the recent paper of Prohorov [6]. We shall cite here some results as an introduction to our theorem, proof of which will be provided in the next section.

Prohorov [5] has given for the general case of arbitrary sequences of independent random variables necessary and sufficient conditions for the validity of the SLLN expressed in terms of probabilities of sums of some segments of the sequence considered. For some particular cases, namely for the case of normal variables X_i $(i = 1, 2, ...)$ and for X_i satisfying the relation

$$|X_i| = o\left(\frac{i}{\log\log i}\right) \quad (i = 1, 2, ...) \tag{1}$$

Prohorov has given necessary and sufficient conditions expressed in terms of variances.

For the general case, there is known Kolmogorov's sufficient condition

$$\sum_{l=1}^{\infty} \frac{D^2(X_i)}{i^2} < \infty$$

which is not necessary but nevertheless, cannot be weakened in the sense that if, for some sequence $\{\sigma_i^2\}$, this condition is not satisfied, it is possible to construct a sequence of independent random variables Y_i with prescribed variances $D^2(Y_i) = \sigma_i^2$ not obeying the SLLN. It can then be easily deduced (although it seems that this has not been explicitly formulated) that, for sequences of arbitrary independent random variables, there do not exist necessary and sufficient conditions for the validity of the SLLN expressed in terms of variances. We shall prove in the next section a stronger result, namely that there do not exist for sequences of random variables satisfying relations (2) necessary and sufficient conditions expressed in variances.

The problem of finding necessary and sufficient conditions for the validity of the SLLN expressed in terms of moments of order higher than 2 will be treated in sections **3** and **4**.

2. Let $\{X_i\}$ be a sequence of symmetric random variables satisfying the inequalities

$$|X_i| < i \quad (i = 1, 2, \ldots). \tag{2}$$

THEOREM 1. *There do not exist for sequences of independent, symmetric random variables satisfying relations* (2) *necessary and sufficient conditions for the validity of the SLLN expressed in terms of the variances* σ_i^2 *of* X_i.

In other words:

if $W(\sigma_1^2, \sigma_2^2, \sigma_3^2, \ldots)$ *is any sufficient condition for the validity of the SLLN, there exists a sequence of independent symmetric random variables satisfying* (2) *obeying the SLLN and not satisfying the condition* W.

Proof. In order to prove the theorem it is sufficient to show that there exist: a sequence of constants $\{\sigma_i^2\}$ and two sequences of symmetric independent random variables $\{X_i\}$ and $\{Y_i\}$, one of which, say, $\{X_i\}$ obeys the SLLN, while the other does not, both satisfying relations (2) and the equalities

$$D^2(X_i) = D^2(Y_i) = \sigma_i^2.$$

We define now the random variables X_i and Y_i as follows:

$$P\left(X_i = -\frac{i}{\log i}\right) = P\left(X_i = \frac{i}{\log i}\right) = \frac{\log i}{2i}, \tag{3}$$

$$P(X_i = 0) = 1 - \frac{\log i}{i},$$

(4)
$$P(Y_i=-\beta i)=P(Y_i=\beta i)=\frac{1}{2\beta^2 i\log i},$$
$$P(Y_i=0)=1-\frac{1}{\beta^2 i\log i},$$
where $0<\beta<1$. We have
$$\sigma^2=\frac{i}{\log i}=D^2(X_i)=D^2(Y_i).$$

The random variables X_i defined by (3) satisfy relation (1). It is now easy to show that the sequence $\{X_i\}$ satisfies for arbitrary $\varepsilon<0$ the relation

(5)
$$\sum_{r=1}^{\infty}\exp\left(-\frac{\varepsilon}{H_r}\right)<\infty,$$
where
$$H_r=\frac{1}{2^{2r}}\sum_{i=2^r+1}^{2^{r+1}}D^2(X_i).$$

Indeed, we have
$$\exp\left(-\frac{\varepsilon}{H_r}\right)<\exp\left(-\frac{\varepsilon r}{2}\log 2\right),$$
and the right side of this inequality is a term of a convergent sequence. In virtue of a theorem of Prohorov [5] the sequence $\{X_i\}$ obeys the SLLN.

On the other hand, the sequence $\{Y_i\}$ does not obey the SLLN. Indeed, if the contrary is true, the relation

(6)
$$P\left(\frac{Y_i}{i}\to 0\right)=1$$
must hold. It follows, however, from (4) that
$$\sum_{i=1}^{\infty}P\left(\frac{|Y_i|}{i}=\beta\right)=\infty,$$
hence, in virtue of the Borel-Cantelli lemma, the events $|Y_i|=\beta i$ occur infinitely often which contradicts relation (6). Consequently, $\{Y_i\}$ does not obey the SLLN.

Theorem 1 is thus proved.

3. We shall consider in this section arbitrary random variables without any restrictions. There are known sufficient conditions for the validity of the SLLN expressed in terms of moments of order higher than 2. We mention here Brunk's generalization of Kolmogorov's condition [1] as well as a set of sufficient conditions obtained recently by Prohorov [6]. We shall deal in this section with the problem of necessary and sufficient conditions for the validity of the SLLN expressed in terms of moments of any finite (arbitrary but fixed) order.

Let $\{X_i\}$ $(i=1,2,\ldots)$ be a sequence of independent random variables. Denote

$$m_{ij}=EX_i^j \quad (i=1,2,\ldots;\ \ j=1,\ldots,r)$$
$$m_i^{(r)}=(m_{i1},\ldots,m_{ir}).$$

We prove the following

THEOREM 2. *Let r be an arbitrary given natural number. There do not exist for arbitrary sequences of independent random variables necessary and sufficient conditions for the validity of the strong law of large numbers expressed in terms of the moments $m_{i1},\ldots m_{ir}$.*

Proof. Let $W(m_1^{(r)},m_2^{(r)},m_3^{(r)},\ldots)$ be a sufficient condition for the validity of the SLLN, provided that the considered moments exist. Let $\{X_i\}$ be a sequence of independent random variables satisfying the condition W. Denote

$$Y_i=\begin{cases} X_i & \text{for} \quad |X_i|\leqslant a_i,\\ \varphi_i(X_i) & \text{for} \quad |X_i|>a_i,\end{cases}$$

where φ_i are Baire functions and a_i is defined by the relation

$$a_i=\inf_a\left\{a: P(|X_i|>a)\leqslant\frac{1}{i^2}\right\}.$$

The sequences $\{X_i\}$ and $\{Y_i\}$ are equivalent in the sense of Khintchine, hence $\{Y_i\}$ obeys the SLLN if $\{X_i\}$ does. On the other hand, since the existence of moments of any order without any additional assumption is not sufficient for the validity of the SLLN, there exists at least one sequence of vectors $\{m_{io}^{(r)}\}=\{m_{i1o},\ldots,m_{iro}\}$ for which the condition W is not satisfied. Now, since we may consider arbitrary random variables, we have at our disposal the choice of the functions φ_i, and it is evident that they can be chosen in such a way that

$$EY_i^j=m_{ijo} \quad (i=1,2,\ldots;\ \ j=1,\ldots,r).$$

The sequence $\{Y_i\}$ will thus not satisfy the condition W.

Theorem 2 is thus proved.

4. Concluding remarks. Remark 1. It does not follow from Theorem 2 that there do not exist necessary and sufficient conditions expressed in terms of moments of any finite order for bounded random variables satisfying (2). If such a condition W exists, then there exist necessary and sufficient conditions for arbitrary random variables X_i expressed in terms of moments of truncated random variables X_i', where

$$X_i'=\begin{cases} X_i & \text{for} \quad |X_i|<i,\\ 0 & \text{for} \quad |X_i|\geqslant i\end{cases}$$

with the additional condition

$$\sum_{i=1}^{\infty} P(|X_i| \geqslant i) < \infty.$$

However, it is the author's conviction that, for random variables satisfying (2), the assertion of Theorem 2 remains true.

Remark 2. For the general case, including unbounded random variables, there is little hope to obtain satisfactory necessary and sufficient conditions, since such conditions must depend on more intrinsic properties of the probability distributions of the random variables considered than their moments of finite order. The existence of necessary and sufficient conditions expressed in terms of moments of the orders 1 up to r is not excluded by Theorem 2, if the moments of orders higher than r are determined by these of orders up to r. Indeed, Prohorov has found necessary and sufficient conditions expressed in terms of moments of order 2 for normal variables.

It seems, therefore, that further advancement in this field will consist in finding necessary and sufficient conditions for comparatively narrow classes of random variables.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] H. D. Brunk, Duke Mathem. Journal, **1** (1948), 181-189.
[2] K. L. Chung, Proceedings of the Second Berkeley Symposium (1951), 341-352.
[3] A. N. Kolmogorov, Comptes Rendus Acad. Sci. Paris, **191** (1930), 910-911.
[4] M. M. Loeve, *Probability theory*, New York (1955).
[5] Yu. V. Prohorov, Izv. Akad. Nauk. SSSR, **14** (1950), 523-536.
[6] — *Theory of probability* (in Russian), Primienienia, **3** (1958), 153-165.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

MATHEMATICS

On Haar Functions and on the Schauder Basis of the Space $C_{\langle 0,1\rangle}$

by

Z. CIESIELSKI

Presented by W. ORLICZ on February 18, 1959

1. We begin by formulating the definition of Haar functions $\{\chi_n(t)\}$. Let

$$\chi_1(t) \equiv 1 \quad \text{in} \quad \langle 0, 1\rangle,$$

and let

$$\chi_{2^n+k}(t) = \begin{cases} \sqrt{2^n} & \text{for} \quad t \in \left\langle \dfrac{2k-2}{2^{n+1}}, \dfrac{2k-1}{2^{n+1}} \right), \\ -\sqrt{2^n} & \text{for} \quad t \in \left(\dfrac{2k-1}{2^{n+1}}, \dfrac{2k}{2^{n+1}} \right\rangle, \\ 0 & \text{otherwise in} \quad \langle 0, 1\rangle, \end{cases}$$

where $n = 0, 1, \ldots$; $k = 1, \ldots, 2^n$. In this section the following notations will be used: $C_{\langle 0,1\rangle}$ denotes the space of all functions $x(t)$ continuous in $\langle 0, 1\rangle$, $\|x\| = \max\limits_{\langle 0,1\rangle} |x(t)|$, $\omega_1(\delta)$ denotes the modulus of continuity of the function $x(\cdot) \in C_{\langle 0,1\rangle}$, i. e

$$\omega_1(\delta) = \sup_{\substack{t_1,t_2 \in \langle 0,1\rangle \\ |t_1-t_2| \leqslant \delta}} |x(t_1) - x(t_2)|;$$

moreover,

$$S_n(t) = \sum_{k=1}^{n} a_k \chi_k(t), \quad \text{where} \quad a_k = \int_0^1 x(t)\chi_k(t)\,dt.$$

The following conditions are satisfied obviously: $\omega_1(0) = 0$, $\omega_1(\delta)$ increases with $\delta \in \langle 0, 1\rangle$ and $\omega_1(\delta_1 + \delta_2) \leqslant \omega_1(\delta_1) + \omega_1(\delta_2)$ for $\delta_1, \delta_2, \delta_1 + \delta_2 \in \langle 0, 1\rangle$. We shall say that the non-negative function $\omega(t)$, defined on $\langle 0, 1\rangle$, satisfies the condition (*), if it satisfies the following one

$$\int_0^\delta \frac{\omega(t)}{t}\,dt \leqslant K\omega(\delta) \quad \text{for} \quad 0 \leqslant \delta \leqslant 1,$$

where K denotes some positive constant. Moreover, $\omega(t)$ will be said to satisfy the conditions (**) if it satisfies the following ones: $\omega(t)$ increases with $t \in \langle 0, 1\rangle$ and $\omega(2t) \leqslant M\omega(t)$ for $0 \leqslant 2t \leqslant 1$, where M denotes some positive constant.

LEMMA 1. *If $x(\cdot) \in C_{\langle 0,1\rangle}$, then the inequality*

$$\sup_{1\leqslant k\leqslant 2^n} |a_{2^n+k}| \leqslant \frac{1}{2\sqrt{2^n}}\omega_1\left(\frac{1}{2^{n+1}}\right)$$

holds for $n = 0, 1, \ldots$

For the proof it suffices to apply the formula

$$a_{2^n+k} = \frac{1}{2\sqrt{2^n}}\int_0^1 \left[x\left(\frac{t}{2^{n+1}}+\frac{2k-2}{2^{n+1}}\right) - x\left(\frac{t}{2^{n+1}}+\frac{2k-1}{2^{n+1}}\right)\right]dt\,.$$

THEOREM 1. *Let $x(\cdot) \in C_{\langle 0,1\rangle}$, and let $\omega_1(\delta)$ satisfy the condition (*). Then we have*

$$\|x - S_n\| \leqslant \frac{2K}{\log 2}\omega_1\left(\frac{1}{n}\right) \quad \textit{for} \quad n \geqslant 2\,.$$

Proof. It is known that, for every $x(\cdot) \in C_{\langle 0,1\rangle}$, the series

$$\sum_{n=1}^{\infty} a_n \chi_n(t)$$

converges to $x(t)$ uniformly in $\langle 0, 1\rangle$ ([1], p. 121). Thus,

$$x(t) - S_n(t) = \sum_{k=n+1}^{\infty} a_k \chi_k(t)\,.$$

Let $2^q \leqslant n < 2^{q+1}$. Then, by Lemma 1 and by our assumptions, we have

$$\begin{aligned}\Big|\sum_{k=n+1}^{\infty} a_k\chi_k(t)\Big| &\leqslant \sum_{p=q}^{\infty}\sum_{k=1}^{2^p} |a_{2^p+k}\chi_{2^p+k}(t)| \\ &\leqslant \sum_{p=q}^{\infty} 2\sqrt{2^p}\sup_{1\leqslant k<2^p}|a_{2^p+k}| \leqslant \sum_{p=q+1}^{\infty}\omega_1\left(\frac{1}{2^p}\right) \\ &\leqslant \frac{K}{\log 2}\omega_1\left(\frac{1}{2^q}\right) \leqslant \frac{2K}{\log 2}\omega_1\left(\frac{1}{n}\right).\end{aligned}$$

THEOREM 2. *Suppose the function $\omega(t)$ satisfies the conditions (**). Moreover, for $x(\cdot) \in C_{\langle 0,1\rangle}$, let the inequality*

$$\|x - S_n\| \leqslant \omega\left(\frac{1}{n}\right)$$

hold for $n = 1, 2, \ldots$, then $\omega_1(\delta) \leqslant 4M\omega(\delta)$ for $0 \leqslant \delta \leqslant 1$.

Proof. Let p and ν denote any non-negative integers such that $1 \leqslant p \leqslant 2^\nu$. Thus, for $t \in \left(\frac{p-1}{2^\nu}, \frac{p}{2^\nu}\right)$ we have $S_{2^\nu}(t) = c_{p\nu}$, where the constant $c_{p\nu}$ depends only on p and ν. Hence, we get

$$S_{2^\nu}(t') - S_{2^\nu}(t'') = 0$$

for $t', t'' \in \left(\frac{p-1}{2^\nu}, \frac{p}{2^\nu}\right)$. Applying the last equality, we have

$$|x(t') - x(t'')| \leqslant |x(t') - S_{2^\nu}(t')| + |x(t'') - S_{2^\nu}(t'')| \leqslant 2\omega\left(\frac{1}{2^\nu}\right);$$

the latter holds uniformly in $1 \leqslant p \leqslant 2^\nu$. By continuity of the function $x(t)$, we have $|x(t') - x(t'')| \leqslant 2\omega\left(\frac{1}{2^\nu}\right)$ for all $t', t'' \in \left\langle\frac{p-1}{2^\nu}, \frac{p}{2^\nu}\right\rangle$. Now let t_1, t_2 denote any points of the interval $\langle 0, 1\rangle$. Evidently, we may assume that $t_1 \neq t_2$. There exists a non-negative integer ν_0 such that

$$\frac{1}{2^{\nu_0}} \geqslant |t_1 - t_2| > \frac{1}{2^{\nu_0+1}}.$$

Moreover, there is a positive integer p_0 such that either t_1 and t_2 both belong to $\left\langle\frac{p_0-1}{2^{\nu_0}}, \frac{p_0}{2^{\nu_0}}\right\rangle$, or $\min(t_1, t_2) \in \left\langle\frac{p_0-1}{2^{\nu_0}}, \frac{p_0}{2^{\nu_0}}\right\rangle$ and $\max(t_2, t_2) \in \left\langle\frac{p_0}{2^{\nu_0}}, \frac{p_0+1}{2^{\nu_0}}\right\rangle$. In the first case we have

$$|x(t_1) - x(t_2)| \leqslant 2\omega\left(\frac{1}{2^{\nu_0}}\right)$$

and in the second case we obtain

$$|x(t_1) - x(t_2)| \leqslant \left|x(t_1) - x\left(\frac{p_0}{2^\nu}\right)\right| + \left|x\left(\frac{p_0}{2^{\nu_0}}\right) - x(t_2)\right| \leqslant 4\omega\left(\frac{1}{2^{\nu_0}}\right).$$

In both cases we have the inequalities

$$|x(t_1) - x(t_2)| \leqslant 4\omega\left(\frac{1}{2^{\nu_0}}\right) \leqslant 4M\omega(|t_1 - t_2|),$$

hence

$$\omega_1(\delta) \leqslant 4M\omega(\delta)$$

and the proof of the theorem is complete.

2. In this section the following notation and definitions will be used in addition to those of the preceding one. We introduce the set of functions $\{\varphi_n(t)\}$, wherein $\varphi_0(t) \equiv 1$, $\varphi_n(t) = \int_0^t \chi_n(\tau)\,d\tau$ for $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$ Moreover, use will be made of the following notations for $x(\cdot) \in C_{\langle 0,1\rangle}$: $T_n(t) = \sum_{k=0}^{n} b_k\varphi_k(t)$, wherein $b_0 = x(0)$, $b_n = \int_0^1 \chi_n(t)\,dx(t)$ for $n = 1, 2, \ldots$;

the function $\omega_2(\delta)$ will denote the modulus of continuity of the second order of the function $x(\cdot)\in C_{\langle 0,1\rangle}$, i. e.

$$\omega_2(\delta) = \sup_{\substack{t_1,t_2\in\langle 0\ 1\rangle \\ |t_1,-t_2|\leqslant\delta}} \left| x(t_1)+x(t_2)-2x\left(\frac{t_1+t_2}{2}\right)\right| .$$

Obviously, $\omega_2(\delta)$ does not decrease in $\langle 0,1\rangle$. Moreover, it is well known that $\omega_2(2\delta)\leqslant 4\omega_2(\delta)$ for $0\leqslant 2\delta\leqslant 1$. This is a trivial consequence of the identity

$$\Delta_{2h}^2 x(t) = \Delta_h^2 x(t) + 2\Delta_h^2 x(t+h) + \Delta_h^2 x(t+2h),$$

wherein

$$\Delta_h^2 x(t) = x(t+2h) - 2x(t+h) + x(t).$$

THEOREM 3. *The set* $\{\varphi_n(t)\}$, $n = 0,1,\ldots$, *constitutes the Schauder basis in the space* $C_{\langle 0,1\rangle}$ (see e. g. [1], p. 50). *Moreover,*

$$x(t) = x(0) + \sum_{n=1}^{\infty}\Big[\int_0^1 \chi_n(\tau)\,dx(\tau)\Big]\varphi_n(t).$$

Proof. Let $r_0=0$, $r_1=1$ and $r_{2^n+k}=(2k-1)/2^{n+1}$ for $n=0,1,\ldots;$ $k=1,\ldots,2^n$. For $n\geqslant 1$, denote by $\sigma_n(t)$ the poligonal line, defined on $\langle 0,1\rangle$, which has its angular points exactly at $r_2,\ldots,r_n$, and which satisfies the conditions $\sigma_n(r_i)=x(r_i)$ for $i=0,\ldots,n$. Let us put $\sigma_0(t)\equiv x(0)$ for $0\leqslant t\leqslant 1$. It is evident that $\|x-\sigma_n\|\to 0$ with $n\to\infty$. Hence, the series

$$\sigma_0(t) + \sum_{n=1}^{\infty}[\sigma_n(t)-\sigma_{n-1}(t)]$$

converges to $x(t)$ uniformly in $\langle 0,1\rangle$. On the other hand, there exist the numbers c_n, $n=1,2,\ldots$, such that

$$\sigma_n(t)-\sigma_{n-1}(t) = c_n\varphi_n(t).$$

Thus,

$$x(t) = x(0)+\sum_{n=1}^{\infty} c_n\varphi_n(t).$$

It can easily be seen that

$$c_1 = x(1)-x(0) = \int_0^1 \chi_1(t)\,dx(t)$$

and

$$c_{2^n+k} = \sqrt{2^n}\left[2x\left(\frac{2k-1}{2^{n+1}}\right)-x\left(\frac{k-1}{2^n}\right)-x\left(\frac{k}{2^n}\right)\right] = \int_0^1 \chi_{2^n+k}(\tau)\,dx(\tau)$$

for $n=0,1,\ldots;$ $k=1,\ldots,2^n$.

As a conclusion from this theorem the representation theorem for the linear functionals defined on $C_{\langle 0,1\rangle}$ may be obtained, but it will be omitted here. Moreover, by Theorem 3 the following one may be proved:

Let $g(t)$ be the function of bounded variation on $\langle 0,1\rangle$. If $x(\cdot)\in C_{\langle 0,1\rangle}$ and $x(0)=0$, then

$$\int_0^1 g(t)\,dx(t)=\lim_{n\to\infty}\int_0^1 g_n(t)\,dx(t),$$

wherein $g_n(t)$ denotes the n-th partial sum of the Fourier-Haar series of $g(t)$.

LEMMA 2. *If $x(\cdot)\in C_{\langle 0,1\rangle}$, then*

$$\sup_{1\leqslant k\leqslant 2^n}|b_{2^n+k}|\leqslant\sqrt{2^n}\,\omega_2\left(\frac{1}{2^n}\right)$$

for $n=0,1,\ldots;\ k=1,\ldots,2^n$.

The obvious proof will be omitted.

THEOREM 4. *If $\omega_2(\delta)$ satisfies the condition $(*)$, then*

$$\|x-T_n\|\leqslant\frac{8K}{\log 2}\,\omega_2\left(\frac{1}{n}\right)$$

for $n=2,3,\ldots$

Proof. Since the series

$$\sum_{n=0}^{\infty} b_n\varphi_n(t)$$

converges to $x(t)$ uniformly in $\langle 0,1\rangle$, we have

$$x(t)-T_n(t)=\sum_{k=n+1}^{\infty} b_k\varphi_k(t).$$

By the last inequality and by Lemma 2 we obtain, for $2^q<n\leqslant 2^{q+1}$,

$$\begin{aligned}|x(t)-T_n(t)| &\leqslant\sum_{p=q}^{\infty}\sum_{k=1}^{2^p}|b_{2^p+k}\varphi_{2^p+k}(t)|\\ &\leqslant\frac{1}{2}\sum_{p=q}^{\infty}\frac{1}{\sqrt{2^p}}\sup_{1\leqslant k<2^p}|b_{2^p+k}|\\ &\leqslant\frac{1}{2}\sum_{p=q}^{\infty}\omega_2\left(\frac{1}{2^p}\right)\leqslant\frac{K}{2\log 2}\,\omega_2\left(\frac{1}{2^{q-1}}\right)\\ &\leqslant\frac{K}{2\log 2}\,\omega_2\left(\frac{4}{n}\right)\leqslant\frac{8K}{\log 2}\,\omega_2\left(\frac{1}{n}\right).\end{aligned}$$

Thus, the theorem is proved.

It is not known whether Theorem 2 remains true if we replace $\{\chi_n(t)\}$ by $\{\varphi_n(t)\}$, $S_n(t)$ by $T_n(t)$ and $\omega_1(\delta)$ by $\omega_2(\delta)$, respectively.

The following two theorems are analogical to that given in [2] for the set $\{\chi_n(t)\}$.

THEOREM 5. *Let $x(\cdot) \epsilon C_{\langle 0,1\rangle}$. If the series*

$$\sum_{n=1}^{\infty} \frac{1}{n} \omega_2\left(\frac{1}{n}\right)$$

converges, then the series

$$\sum_{n=1}^{\infty} |b_n \varphi_n(t)|$$

is uniformly convergent in $\langle 0, 1\rangle$.

Proof. By Lemma 2 we have

$$\sum_{n=2}^{\infty} |b_n \varphi_n(t)| = \sum_{n=0}^{\infty} \sum_{k=1}^{2^n} |b_{2^n+k} \varphi_{2^n+k}(t)| \leqslant \frac{1}{2} \sum_{n=0}^{\infty} \omega_2\left(\frac{1}{2^n}\right) \leqslant \sum_{n=1}^{\infty} \frac{1}{n} \omega_2\left(\frac{1}{n}\right).$$

THEOREM 6. *Let $x(\cdot) \epsilon C_{\langle 0,1\rangle}$. If the series*

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \omega_2\left(\frac{1}{n}\right)$$

converges, then

$$\sum_{n=1}^{\infty} \frac{|b_n|}{n} < \infty.$$

Proof. By Lemma 2 we have

$$|b_n| \leqslant 4\sqrt{n}\, \omega_2\left(\frac{1}{n}\right).$$

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT MATEMATYCZNY PAN)

REFERENCES

[1] S. Kaczmarz und H. Steinhaus, *Theorie der Orthogonalreihen*, Warsaw-Lwów, 1935.

[2] Z. Ciesielski and J. Musielak, *On absolute convergence of Haar series*, Coll. Math., 7 (1959).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

THEORETICAL PHYSICS

On the Deformability of the Nuclear Core

by

Z. SZYMAŃSKI

Presented by L. INFELD on January 29, 1959

It is well known that the quadrupole moment of the ^{17}O nucleus with a single external neutron is relatively large ($Q \approx -0.04\ 10^{-24}\ cm^2$) and the half-life of its first excited level is relatively low ($T_{1/2} \approx 2.5\ 10^{-10}$ sec). Both these facts are incompatible with the shell model, if only a simple configuration of one neutron moving in the spherical rigid potential well is introduced. Similar discrepancies were observed for several other nuclei which contain only one nucleon outside a double closed shell core, for instance ^{17}F, ^{207}Pb, etc. [1]-[9].

A possible explanation of these effects is the assumption of the deformability of the nuclear core by the external nucleon [1]. According to this assumption the contribution to the total mass-quadrupole moment is supplied not only by the single nucleon, but also by the core deformation induced by the orbital motion of the single nucleon. Let us assume that core deformation is very weak, so that it may be treated as a perturbational parameter. We may hope that, for nuclei of this kind, it is a good approximation. Then, writing the total mass-quadrupole moment in the form of a sum:

$$Q_M = Q_{\text{core}} + Q_p\,, \tag{1}$$

(where Q_p — is the quadrupole moment of the last odd particle), we can assume that

$$Q_{\text{core}} = \gamma \cdot Q_p\,, \tag{2}$$

Here, the coefficient γ so defined, can be called the deformability of the nuclear core.

The charge quadrupole moment can then be expressed by the formula

$$Q_e = e_p Q_p + \frac{eZ}{A} \cdot \gamma \cdot Q_p\,,$$

where e_p denotes the charge of the last nucleon ($e_p = 0$ for a neutron, $e_p = e$ for a proton). Thus, the effect of the deformability of the core can be expressed by introducing the concept of the effective charge:

$$e_{\text{eff}} = e_p + \frac{e \cdot Z}{A} \cdot \gamma \, .$$

Assuming the nuclear potential in the harmonic oscillator form, it can be shown [1] that the resulting deformability equals 1. Such a value corresponds well to the experimental data for light nuclei. For instance, the effective charge for ^{17}F and ^{17}O is $(\frac{3}{2})e$ and $(\frac{1}{2})e$, respectively. For heavy nuclei, however, the value $\gamma = 1$ is not in agreement with experimental data. Indeed, in the neighbourhood of the ^{207}Pb nucleus, for example, the effective charge which can be computed from the observed [10] lifetimes is [9]:

$$e_{\text{eff}} \approx 1.15\, e$$

which corresponds to the value

(3) $$\gamma = 1.15\, A/Z = 2.91 \, .$$

This discrepancy, however, may be predicted, for it is well known that real nuclear potential is essentially different from the harmonic oscillator potential in the domain of heavy nuclei. The present paper is concerned with the calculation of the deformability γ, when the other extremity-square well potential is assumed. Since the real potential is probably intermediate between the harmonic oscillator and the square well, it may be presumed that the numerical results obtained for γ will be overestimated. Let us assume that the nuclear core is deformed in a selfconstistent way, i. e. that the nuclear shape is determined by the wave function which coincides with the potential well shape. Using Moszkowski's [11] wave functions for the spheroideal infinite potential well we can calculate the selfconsistent deformation β. It turns out that the deformation β is proportional to the quadrupole moment Q_p of the last particle. The coefficient of proportionality depends only on the nuclear core. If Q core is then calculated, we obtain an expression of type (2) in which the deformability γ may be written in the final form

(4) $$\gamma = -1 + \frac{5}{12} \cdot \frac{\sum\limits_{nl} (2l+1)\, R_{nl}^{nl}}{\sum\limits_{nl} \sum\limits_{l'=l\pm 2} a_{e'} \frac{\omega_{ne}\,\omega_{ne'}}{(\omega_{nl}^2 - \omega_{nl'}^2)^2} R_{nl}^{nl'}} \, ,$$

where $\sum\limits_{nl}$ denotes the sum over quantum numbers n, l from closed shells, ω_{nl} — is an n-th node of $j_l(x)$, and coefficients $a_{l'}$, are

$$a_{l'} = \begin{cases} -(l'+1)(l'+2) & \text{for} \quad l' = l-2 \, , \\ l'(l'-1) & \text{for} \quad l' = l+2 \, . \end{cases}$$

Finally, radial integrals are defined by the following expressions:

$$R_{nl}^{n'l'} = (R_{nl} r^2 R_{n'l'}) . \tag{5}$$

For the numerical estimation of the formula (4) radial integrals were assumed to be all equal, i. e. $R_{nl}^{nl'} \approx R_{nl}^{nl} \approx R$. In this case the results of the numerical computations are given in Table I.

TABLE I

Configuration	Number of particles in given shell	Total number of particles	γ
1 *s*	4	4	5.26
1 *p*	12	16	4.65
1 *d*	20	36	4.14
2 *s*	4	40	4.26
1 *f*	28	68	3.93
2 *p*	12	80	4.09
1 *g*	36	116	3.85
2 *d*	20	136	3.95
1 *h*	44	180	3.63
3 *s*	4	184	3.64
2 *f* (neutrons only)	14	198	3.70
1 *i* (neutrons only)	26	224	3.63
3 *p* (neutrons only)	6	230	3.58

Discussion

It is seen from Table I that the deformability calculated for the square-well potential is larger than that for the harmonic oscillator. The latter is thus more resistant to deformation. The square-well deformability γ decreases on the average for the increasing nucleon number. For heavy nuclei ($A \sim 200$) the calculated deformability γ (see Table I) is somewhat larger than the experimental value given by (3). This is connected with the fact that the real nuclear potential deviates from the pure square-well shape.

The author wishes to express his sincere thanks to Dr. B. Mottelson for suggesting this problem and for his many valuable remarks. He is also indebted to Dr. B. Bes for a helpful discussion.

The calculation was partly carried out during the author's stay at the Summer School of Theoretical Physics in Les Houches and he takes this opportunity to acknowledge his gratitude to the School Board for making his stay possible.

INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WARSZAWSKIEGO)
INSTITUTE FOR NUCLEAR RESEARCH, WARSAW
(INSTYTUT BADAŃ JĄDROWYCH, WARSZAWA)

REFERENCES

[1] B. Mottelson, *Lectures on the collective motion of nuclei given at Les Houches* (1958) — to be published.
[2] F. C. Baker, Phil. Mag., **1** (1956), 329.
[3] W. W. True, Phys. Rev., **101** (1956), 1342.
[4] B. J. Raz, ibid., **107** (1957), 1201.
[5] M. A. Perks, Proc. Phys. Soc., A **68** (1955), 1083.
[6] H. Horie, A. Arima, Phys. Rev., **99** (1955), 778.
[7] R. D. Amado, R. J. Blin-Stoyle, Proc. Phys. Soc., A **70** (1957), 532.
[8] R. D. Amado, Phys. Rev., **108** (1957), 1462.
[9] W. W. True, K. Ford, ibid., **109** (1958), 1675.
[10] N. H. Lazar, E. D. Klema, ibid., **98** (1955), 710.
[11] S. Moszkowski, ibid., **99** (1955), 803.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

THEORETICAL PHYSICS

A Fixed-Source Approach to Scattering of Kaons

by
W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on February 2, 1959

Introduction

In this paper a fixed-source approach of the Chew-Low type [1], [2] is applied to the scattering of K mesons on nucleons (cf. [3]). The Hamiltonian used is that proposed in [4]. The formalism introduced in [4] makes it possible to treat all baryons (N, Λ^0, Σ, Ξ) as states of one baryon-particle (B), just like the isospin formalism makes it possible to describe, e. g. proton (p) and neutron (n), as isospin states of one nucleon-particle (N). In the present paper, this baryon-particle is represented by a fixed source with some internal degrees of freedom (corresponding to the spin, isospin and strangeness of the baryon).

We assume the Hamiltonian in the form

$$H = H^B + H^\pi + H^K + H^{B\pi} + H^{BK}, \tag{1}$$

where

$$H^K = \sum_{\vec{k}} (a^*_{\vec{k}a} a_{\vec{k}a} + b^*_{\vec{k}a} b_{\vec{k}a})\,\omega_k, \qquad \omega_k = \sqrt{\vec{k}^2 + \mu_K} \tag{2}$$

and

$$H^{BK} = \frac{f^0}{\mu_K}\sigma_i \xi_a \int d_3x \varrho(|\vec{x}|)\partial_i \big(CK(\vec{x})\big)_a + h.\,c. \\ + \lambda \Big[\int d_3x \varrho(|\vec{x}|)\,\xi_a \big(CK(\vec{x})\big)_a + h.\,c.\Big]^2. \tag{3}$$

Here, we have $C = -\tau_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, $K = (K_a) = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$, $K^* = (K^*_a) = \begin{pmatrix} K^- \\ \overline{K^0} \end{pmatrix}$ and

$$K_a(\vec{x}) = \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} (a_{\vec{k}a} e^{i\vec{k}\vec{x}} + b^*_{\vec{k}a} e^{-i\vec{k}\vec{x}}). \tag{4}$$

For further explanation of the notation used compare paper [4].

We assume with regard to the Hamiltonian (1) that

$$H|N\rangle = 0\,, \quad H|Y\rangle = \Delta M_Y |Y\rangle \quad (\Delta M_Y = M_Y - M_N)\,, \tag{5}$$

where $|N\rangle$ and $|Y\rangle$ are states of the real nucleon (N) and real hyperons ($Y = \Lambda^0, \Sigma, \Xi$), respectively. The explicit form of the interaction $H^{B\pi}$ will not be needed. We denote the real baryon states $|B\rangle = |N\rangle, |Y\rangle$ by Φ_σ, where σ stands for spin, isospin and strangeness quantum numbers of the baryon. All eigenstates of the Hamiltonian (1) are scattering states and have the form

$$\Phi^{\pm}_\lambda = |\lambda\rangle + \frac{1}{E_\lambda - H \pm i\epsilon}(H - E_\lambda)|\lambda\rangle, \quad (E_\lambda - H)\Phi^{\pm}_\lambda = 0\,. \tag{6}$$

Especially for $\lambda = \sigma$ we have $\Phi^{\pm}_\sigma = \Phi_\sigma$, $E_\sigma = \Delta M_\sigma$, for $\lambda = (\sigma\vec{k}a)$

(7)

$$\Phi^{\pm}_{\sigma\vec{k}a} = a^*_{\vec{k}a}\Phi_\sigma + \frac{1}{\Delta M_\sigma + \omega_k - H \pm i\epsilon}(H - \Delta M_\sigma - \omega_k) a^*_{\vec{k}a}\Phi_\sigma\,, \quad E_{\sigma\vec{k}a} = \Delta M_\sigma + \omega_k$$

and for $\lambda = (\sigma\vec{k}\tilde{a})$

($\tilde{7}$)

$$\Phi^{\pm}_{\sigma\vec{k}\tilde{a}} = b^*_{\vec{k}a}\Phi_\sigma + \frac{1}{\Delta M_\sigma + \omega_k - H \pm i\epsilon}(H - \Delta M_\sigma - \omega_k) b^*_{\vec{k}a}\Phi_\sigma\,, \quad E_{\sigma\vec{k}\tilde{a}} = \Delta M_\sigma + \omega_k\,.$$

Elements of the S matrix describing the transitions $\sigma\vec{k}a \to \lambda$ and $\sigma\vec{k}\tilde{a} \to \lambda$ are given obviously by

$$\langle\lambda|S|\sigma\vec{k}a\rangle = \delta_{\lambda,\sigma\vec{k}a} - 2\pi i\,\delta(E_\lambda - \Delta M_\sigma - \omega_k)\,T_{\sigma\vec{k}a}(\lambda) \tag{8}$$

and

$$\langle\lambda|S|\sigma\vec{k}\tilde{a}\rangle = \delta_{\lambda,\sigma\vec{k}\tilde{a}} - 2\pi i\,\delta(E_\lambda - \Delta M_\sigma - \omega_k)\,T_{\sigma\vec{k}\tilde{a}}(\lambda)\,, \tag{$\tilde{8}$}$$

where

$$T_{\sigma\vec{k}a}(\lambda) = \langle\Phi^-_\lambda V_{\vec{k}a}\Phi_\sigma\rangle\,, \quad V_{\vec{k}a} = [H^{BK}, a^*_{\vec{k}a}] \tag{9}$$

and

$$T_{\sigma\vec{k}a}(\lambda) = \langle\Phi^-_\lambda V_{\vec{k}\tilde{a}}\Phi_\sigma\rangle, \quad V_{\vec{k}\tilde{a}} = [H^{BK}, b^*_{\vec{k}a}]\,. \tag{$\tilde{9}$}$$

Using (3) we get

$$V_{\vec{k}a} = \frac{v(k)}{\sqrt{2\omega_k}}\left[\frac{f^0}{\mu_K} i\vec{\sigma}\cdot\vec{k}(\xi C)_a + \lambda \sum_{\vec{k}'} \frac{v(k')}{\sqrt{2\omega_{k'}}}(a^*_{\vec{k}'a} + b_{-\vec{k}'a})\right] = V^*_{-\vec{k}\tilde{a}}\,, \tag{10}$$

where

$$\varrho(|\vec{x}|) = \frac{1}{(2\pi)^3}\sum_{\vec{k}} v(k)\, e^{i\vec{k}\vec{x}}, \quad k = |\vec{k}|\,. \tag{11}$$

Low's equations

We obtain from Eqs. (9) and ($\tilde{9}$) the following Low equations:

$$(12)\quad T_{\sigma\vec{k}a}(\sigma'\vec{k}'a') = \langle\Phi_{\sigma'}[a_{\vec{k}'a'}, V_{\vec{k}a}]\Phi_\sigma\rangle + \sum_\lambda\left[\frac{T^*_{\sigma'\vec{k}'a'}(\lambda)\,T_{\sigma\vec{k}a}(\lambda)}{\Delta M_{\sigma'}+\omega_{k'}-E_\lambda+i\epsilon}+\frac{T^*_{\sigma'-\vec{k}\tilde{a}}(\lambda)\,T_{\sigma-\vec{k}'\tilde{a}'}(\lambda)}{\Delta M_{\sigma'}-\omega_{k'}-E_\lambda}\right]$$

and

$$(\tilde{12})\quad T_{\sigma\vec{k}\tilde{a}}(\sigma'\vec{k}'\tilde{a}') = \langle\Phi_{\sigma'}[b_{\vec{k}'a'}, V_{\vec{k}\tilde{a}}]\Phi_\sigma\rangle + \sum_\lambda\left[\frac{T^*_{\sigma'\vec{k}'\tilde{a}'}(\lambda)\,T_{\sigma\vec{k}\tilde{a}}(\lambda)}{\Delta M_{\sigma'}+\omega_{k'}-E_\lambda+i\epsilon}+\frac{T^*_{\sigma'-\vec{k}a}(\lambda)\,T^*_{\sigma-\vec{k}'a'}(\lambda)}{\Delta M_{\sigma'}-\omega_{k'}-E_\lambda}\right].$$

The scattering amplitudes $T_{\sigma\vec{k}a}(\sigma'\vec{k}'a')$ and $T_{\sigma\vec{k}\tilde{a}}(\sigma'\vec{k}'\tilde{a})$ describe the processes $B+K\to B+K$ and $B+\bar{K}\to B+\bar{K}$, respectively.

We note that, by (10),

$$(13)\qquad \langle\Phi_{\sigma'}V_{\vec{k}a}\Phi_\sigma\rangle = Z^{1/2}_{\sigma'\sigma}\langle u_{\sigma'}V_{\vec{k}a}u_\sigma\rangle = \frac{f_{\sigma'\sigma}}{\mu_K}\frac{v(k)}{\sqrt{2\omega_k}}\,i\vec{k}[\vec{\sigma}\,(\xi C)_a]_{\sigma'\sigma}\,,$$

where the u_σ are states of the bare baryon, $Z_{\sigma'\sigma}$ represent some renormalization constants and $f_{\sigma'\sigma}=Z^{1/2}_{\sigma'\sigma}f^0$ are the renormalized coupling constants.

The Born term in the Eq. (12) has the form

$$(14)\quad T^B_{\sigma\vec{k}a}(\sigma'\vec{k}'a') = \langle\Phi_{\sigma'}[a_{\vec{k}'a'}, V_{\vec{k}a}]\Phi_\sigma\rangle + \sum_{\sigma''}\left[\frac{T^*_{\sigma'\vec{k}'a'}(\sigma'')\,T_{\sigma\vec{k}a}(\sigma'')}{\Delta M_{\sigma'}-\Delta M_{\sigma''}+\omega_{k'}+i\epsilon}+\frac{T^*_{\sigma'-\vec{k}\tilde{a}}(\sigma'')\,T^*_{\sigma-\vec{k}'\tilde{a}'}(\sigma'')}{\Delta M_{\sigma'}-\Delta M_{\sigma''}+\omega_{k'}}\right].$$

In the following considerations we shall confine our attention to the case when σ and σ' correspond to the nucleon states, i. e. we shall consider the processes $N+K\to N+K$ and $N+\bar{K}\to N+\bar{K}$. Then, by virtue of (10) and (13)

$$(15)\quad T^B_{\sigma\vec{k}a}(\sigma'\vec{k}'a') = \frac{v(k')v(k)}{\sqrt{4\omega_{k'}\omega_k}}$$

$$\times\left\{\lambda\delta_{\sigma'\sigma}\delta_{a'a}-\frac{1}{2\mu_K^2}\left(\frac{f^2_{N\Lambda}}{\Delta M_\Lambda+\omega_{k'}}\delta_{\sigma'a}\delta_{a'\sigma}+\frac{f^2_{N\Sigma}}{\Delta M_\Sigma+\omega_{k'}}\tau_{i\sigma'a}\tau_{ia'\sigma}\right)[(\vec{\sigma}\cdot k)(\vec{\sigma}\cdot\vec{k}')]_{\sigma'\sigma}\right\}.$$

We obtain likewise the following formulae for the Born term in the Eq. $(\widetilde{12})$

$$(\widetilde{15}) \quad T^{B}_{\sigma\vec{k}\tilde{a}}(\sigma'\vec{k}'\tilde{a}') = \frac{v(k')\,v(k)}{\sqrt{4\omega_{k'}\omega_k}}$$

$$\times\left\{\lambda\delta_{\sigma'\sigma}\delta_{a'a} - \frac{1}{2\mu_K^2}\left(\frac{f_{N\Lambda}^2}{\Delta M_\Lambda - \omega_{k'}}\,\delta_{\sigma'a'}\delta_{a\sigma} + \frac{f_{N\Sigma}^2}{\Delta M_\Sigma - \omega_{k'}}\,\tau_{i\sigma'a'}\tau_{ia\sigma}\right)[(\vec{\sigma}\cdot\vec{k}')(\vec{\sigma}\cdot\vec{k})]_{\sigma'\sigma}\right\}.$$

We denote by $T_T(\vec{k}',\vec{k})$ and $\widetilde{T}_T(\vec{k}',\vec{k})$ the scattering amplitudes for K and $\bar{K}$ mesons respectively, corresponding to the total isospin T $(T=0,1)$. We can separate the scattering amplitudes for different isospins $T(T=0,1)$ using the following relations (the spin quantum number of the nucleon contained in the index σ is here suppressed)

$$(16) \quad T_1(\vec{k}',\vec{k}) = T_{1\vec{k}1}(1\vec{k}'1) = T_{2\vec{k}2}(2\vec{k}'2)$$
$$= T_{2\vec{k}1}(2\vec{k}'1) + T_{2\vec{k}1}(1\vec{k}'2) = T_{1\vec{k}2}(1\vec{k}'2) + T_{1\vec{k}2}(2\vec{k}1),$$
$$T_0(\vec{k}',\vec{k}) = T_{2\vec{k}1}(2\vec{k}'1) - T_{2\vec{k}1}(1\vec{k}'2) = T_{1\vec{k}2}(1\vec{k}'2) - T_{1\vec{k}2}(2\vec{k}1)$$

and

$$(\widetilde{16}) \quad \widetilde{T}_1(\vec{k}',\vec{k}) = T_{2\vec{k}\tilde{1}}(2\vec{k}'\tilde{1}) = T_{1\vec{k}\tilde{2}}(1\vec{k}'\tilde{2})$$
$$= T_{1\vec{k}\tilde{1}}(1\vec{k}'\tilde{1}) + T_{1\vec{k}\tilde{1}}(2\vec{k}'\tilde{2}) = T_{2\vec{k}\tilde{2}}(2\vec{k}'\tilde{2}) + T_{2\vec{k}\tilde{2}}(1\vec{k}'\tilde{1}),$$
$$\widetilde{T}_0(\vec{k}',\vec{k}) = T_{1\vec{k}\tilde{1}}(1\vec{k}'\tilde{1}) - T_{1\vec{k}\tilde{1}}(2\vec{k}'\tilde{2}) = T_{2\vec{k}\tilde{2}}(2\vec{k}'\tilde{2}) - T_{2\vec{k}\tilde{2}}(1\vec{k}'\tilde{1});$$

$$N = \begin{pmatrix} p \\ n \end{pmatrix},\ K = \begin{pmatrix} K^+ \\ K^0 \end{pmatrix},\ \bar{K} = \begin{pmatrix} K^- \\ \overline{K^0} \end{pmatrix}.$$

The separation of scattering amplitudes for different orbital momenta L $(L=0,1)$ may be carried out by means of the formulae

$$(17) \qquad T_T(\vec{k}',\vec{k}) = -4\pi\frac{v(k')\,v(k)}{\sqrt{4\omega_{k'}\omega_k}}\left[g_T(\omega_k) + \sum_J P_J(\vec{k}',\vec{k})\,h_{JT}(\omega_k)\right]$$

and

$$(\widetilde{17}) \qquad \widetilde{T}_T(\vec{k}',\vec{k}) = -4\pi\frac{v(k')\,v(k)}{\sqrt{4\omega_{k'}\omega_k}}\left[\widetilde{g}_T(\omega_k) + \sum_J P_J(\vec{k}',\vec{k})\,\widetilde{h}_{JT}(\omega_k)\right],$$

where $(J=\frac{1}{2},\frac{3}{2})$

$$(18) \quad P_{1/2}(\vec{k}',\vec{k}) = (\vec{\sigma}\cdot\vec{k}')(\vec{\sigma}\cdot\vec{k}), \qquad P_{3/2}(\vec{k}',\vec{k}) = 3\vec{k}'\cdot\vec{k} - (\vec{\sigma}\cdot\vec{k}')(\vec{\sigma}\cdot\vec{k}).$$

Here, the g's and h's describe the S and P wave for K mesons the $\widetilde{g}$'s and $\widetilde{h}$'s – for $\bar{K}$ mesons. Note that

$$(19) \qquad \sum_{J'} P_{J'}(\vec{k}',\vec{k})\,A_{J'J} = P_J(\vec{k}',\vec{k}), \qquad A = (A_{J'J}) = \tfrac{1}{3}\begin{pmatrix} -1 & 4 \\ 2 & 1 \end{pmatrix}.$$

Fixed-source dispersion relations

We may write the following dispersion relations, if we carry out in (12) and ($\widetilde{12}$) the separation of different partial waves,

$$g_T(\omega) = g_T^B(\omega) + \frac{1}{(2\pi)^2} \int_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')} \left[\frac{\sigma_T^S(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\widetilde{\sigma}_T^S(\omega')}{\omega' + \omega} \right],$$

(20)

$$h_{JT}(\omega) = h_{JT}^B(\omega) + \frac{1}{(2\pi)^2} \int_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{1}{k' v^2(k')} \left[\frac{\sigma_{JT}^P(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\sum_{J'} A_{JJ'} \widetilde{\sigma}_{J'T}^P(\omega')}{\omega' + \omega} \right]$$

and

$$\widetilde{g}_T(\omega) = \widetilde{g}_T^B(\omega) + \frac{1}{(2\pi)^2} \int_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{k'}{v^2(k')} \left[\frac{\widetilde{\sigma}_T^S(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\sigma_T^S(\omega')}{\omega' + \omega} \right],$$

($\widetilde{20}$)

$$\widetilde{h}_{JT}(\omega) = \widetilde{h}_{JT}^B(\omega) + \frac{1}{(2\pi)^2} \int_{\Delta M_\Lambda+\mu_\pi}^{\infty} d\omega' \frac{1}{k' v^2(k')} \left[\frac{\widetilde{\sigma}_{JT}^P(\omega')}{\omega' - (\omega + i\epsilon)} + \frac{\sum_{J'} A_{JJ'} \sigma_{J'T}^P(\omega')}{\omega' + \omega} \right].$$

Here, the σ's and $\widetilde{\sigma}$'s are the total cross-sections (continued to the unphysical region $\langle \Delta M_\Lambda + \mu_\pi, \mu_K \rangle$) for K and $\bar{K}$ mesons, respectively. The Born terms have the forms

$$(21) \quad \left\{ \begin{aligned} & g_T^B(\omega) = -\frac{\lambda}{4\pi} (T = 0, 1), \\ & h_{J0}^B(\omega) = \frac{1}{3} \begin{Bmatrix} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{Bmatrix} \frac{1}{4\pi\mu_K^2} \frac{1}{2} \left(-\frac{f_{N\Lambda}^2}{\Delta M_\Lambda + \omega} + \frac{3 f_{N\Sigma}^2}{\Delta M_\Sigma + \omega} \right), \\ & h_{J1}^B(\omega) = \frac{1}{3} \begin{Bmatrix} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{Bmatrix} \frac{1}{4\pi\mu_K^2} \frac{1}{2} \left(\frac{f_{N\Lambda}^2}{\Delta M_\Lambda + \omega} + \frac{f_{N\Sigma}^2}{\Delta M_\Sigma + \omega} \right) \end{aligned} \right.$$

and

$$(\widetilde{21}) \quad \left\{ \begin{aligned} & \widetilde{g}_T^B(\omega) = -\frac{\lambda}{4\pi} (T = 0, 1), \\ & \widetilde{h}_{JT}^B(\omega) = \begin{Bmatrix} 1 & J = \frac{1}{2} \\ 0 & J = \frac{3}{2} \end{Bmatrix} \frac{1}{4\pi\mu_K^2} \frac{f_{N\Sigma}^2}{\Delta M_\Sigma - \omega} (T = 0, 1). \end{aligned} \right.$$

We can suppose that the charge exchange scattering of K and $\bar{K}$ mesons is small in the present theory, because of $\widetilde{g}_0^B = \widetilde{g}_1^B$, $\widetilde{h}_{J0}^B = \widetilde{h}_{J1}^B$ and

$g_0^B \approx g_1^B$, $h_{J0}^B \approx h_{J1}^B$ (the two last following from the fact that $\Delta M_\Lambda \approx \Delta M_\Sigma$, $f_{N\Lambda} \approx f_{N\Sigma}$). If we use the approximation $\Delta M_\Lambda = \Delta M_\Sigma$ (denote it by ΔM) and $f_{N\Lambda} = f_{N\Sigma}$ (denote it by f_K), then we obtain

$$(22) \qquad h_{JT}^B(\omega) = \frac{\lambda_J}{\Delta M + \omega}, \qquad \lambda_J = \frac{f_K^2}{4\pi\mu_K}\frac{1}{3}\begin{cases} -1 & J = \frac{1}{2} \\ 2 & J = \frac{3}{2} \end{cases} \qquad (T = 0, 1)$$

and

$$(\widetilde{22}) \qquad \widetilde{h}_{JT}^B(\omega) = \frac{\widetilde{\lambda}_J}{\Delta M - \omega}, \qquad \widetilde{\lambda}_J = \frac{f_K^2}{4\pi\mu_K}\begin{cases} 1 & J = \frac{1}{2} \\ 0 & J = \frac{3}{2} \end{cases} \qquad (T = 0, 1).$$

Thus, in this approximation, the charge exchange scattering of K and $\overline{K}$ mesons does not appear completely (it follows also from [4]).

The assumed interaction Hamiltonian (3) does not lead to different masses and coupling constants for Λ^0 and Σ hyperons. Thus, the mass difference $M_\Sigma - M_\Lambda$ may be either introduced phenomenologically (non-field mass) or originated by an interaction Hamiltonian $H^{B\pi}$ (the interaction $H^{B\pi}$ proposed in [4] has such a property). The difference between the coupling constants $g_{N\Lambda}$ and $g_{N\Sigma}$ can be created only by the interaction $H^{B\pi}$. The approximation $\Delta M_\Lambda = \Delta M_\Sigma$ and $f_{N\Lambda} = f_{N\Sigma}$ is, therefore, equivalent in the present theory (cf. also [4]) to the neglection of the baryon-pion interaction (and the dropping of phenomenological corrections).

It follows from (20) that

$$(23) \qquad \operatorname{Im} g_T(\omega) = \frac{k}{4\pi v^2(k)}\sigma_{JT}^S(\omega), \qquad \operatorname{Im} h_{JT}(\omega) = \frac{1}{4\pi k v^2(k)}\sigma_{JT}^P(\omega).$$

The scattering amplitudes g's and h's are connected with phase shifts δ's (being in general complex) by the usual formulae

$$(24) \qquad \begin{aligned} g_T(\omega) &= \frac{1}{kv^2(k)} e^{i\delta_T^S(\omega)} \sin\delta_T^S(\omega), \\ h_{JT}(\omega) &= \frac{1}{k^3v^2(k)} e^{i\delta_{JT}^P(\omega)} \sin\delta_{JT}^P(\omega). \end{aligned}$$

The connection between the σ's and δ's following from (23) and (24) has the form

$$(25) \qquad \sigma(\omega) = \frac{4\pi}{k^2} e^{-2\operatorname{Im}\delta(\omega)} \sin^2 \operatorname{Re}\delta(\omega) + \frac{2\pi}{k^2}\left(1 - e^{-2\operatorname{Im}\delta(\omega)}\right).$$

Relations similar to (23), (24) and (25) are satisfied also for the $\overline{K}$ mesons.

Effective range approximation

In the case $\operatorname{Im}\delta(\omega) = 0$, we obtain from (25) and (24) the following well-known formulae

$$\cot \delta_T^S(\omega) = \frac{1}{kv(k)} \operatorname{Re} \frac{1}{g_T(\omega)} \,, \tag{26}$$
$$\cot \delta_{JT}^P(\omega) = \frac{1}{k^2 v(k)} \operatorname{Re} \frac{1}{h_{JT}(\omega)} \,.$$

Repeating for the scattering of K mesons the effective range approach of Chew and Low, we write for P waves

$$k^3 v(k) h_{JT}^B(\omega^*) \cot \delta_{JT}^P(\omega^*) = \operatorname{Re} \frac{h_{JT}^B(\omega^*)}{h_{JT}(\omega^*)} = 1 - r_{JT}^P \quad (\varDelta M + \omega^*) + \dots \,, \tag{27}$$

where $\omega^* = k^2/2M_N + \omega = k^2/2M_N + \sqrt{k^2 + \mu_K^2}$ is the energy of the nucleon and meson in the centre-of-mass system and

$$r_{JT}^P = \frac{d}{d\omega} \frac{h_{JT}^P(\omega)}{h_{JT}(\omega)} \bigg|_{\varDelta M + \omega = 0} \tag{28}$$
$$= \frac{1}{\lambda_J} \frac{1}{(2\pi)^2} \int_{\varDelta N + \mu_\pi}^{\infty} d\omega' \frac{1}{k'\omega' v^2(k')} \Big[\sigma_{JT}^P(\omega') + \sum_{J'} A_{JJ'} \tilde{\sigma}_{JT}^P(\omega') \Big] .$$

Hence,

$$r_{1/2T}^P = -3 \frac{4\pi \mu_K^2}{f_K^2} \frac{1}{(2\pi)^2} \int_{\varDelta M + \mu_\pi}^{\infty} d\omega' \frac{1}{k'\omega' v^2(k')} [\sigma_{1/2T}^P(\omega') - \tilde{\sigma}_{1/2T}^P(\omega') + 4\tilde{\sigma}_{3/2T}^P(\omega')] \,, \tag{29}$$
$$r_{3/2T}^P = \frac{3}{2} \frac{4\pi \mu_K^2}{f_K^2} \frac{1}{(2\pi)^2} \int_{\varDelta M + \mu_\pi}^{\infty} d\omega' \frac{1}{k'\omega' v^2(k')} [\sigma_{3/2T}^P(\omega') + 2\tilde{\sigma}_{1/2T}^P(\omega') + \tilde{\sigma}_{3/2T}^P(\omega')] \,.$$

We can see that $r_{3/2T}^P > 0$ and, therefore, there appears in our approach a resonance $P_{3/2}$ for the scattering $N + K \to N + K$ in the states $T = 0, 1$ at the energy $\omega_T^* = 1/r_{3/2T}^P - \varDelta M_\varLambda$.

For a rough estimate of the effective range $r_{3/2T}^P$ we write (like in the pion theory)

$$r_{3/2T}^P \sim \frac{f_K^2}{4\pi\mu_K^2} \omega_{\text{cut-off}} \sim \frac{f_K^2}{4\pi\mu_K^2} M_N = \frac{G_K^2}{4\pi} \frac{1}{4M_N} \,,$$

where $\omega_{\text{cut-off}} \sim M_N$ is a mean cut-off energy, $G_K = (2M_N/\mu_K) f_K$ and $G_K^2/4\pi \sim 1$ to 4. Thus,

$$r_{3/2T}^P \sim (\tfrac{1}{4} \text{ to } 1) \frac{1}{M_N} \quad \text{and} \quad \omega_T^* \sim (4 \text{ to } 1) M_N \,.$$

Hence, the hypothetical resonance $P_{3/2}$ for the scattering $N+K \to N+K$ in the states $T=0,1$ would lie in a considerably high energy region.

The considerations contained in this paper should be treated only as a rough estimate of the real situation, because of the fixed-source simplifying assumptions and the effective-range approximation used in the argument.

INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WARSZAWSKIEGO)
INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH PAN)

REFERENCES

[1] G. F. Chew and F. E. Low, Phys. Rev., **101** (1956), 1570.
[2] A. Klein, Phys. Rev., **104** (1956), 1131, 1136.
[3] D. Amati and B. Vitale, Nuovo Cimento, **6** (1957), 1013, 1273.
[4] W. Królikowski, Nuclear Physics, **8** (1958), 461.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

THEORETICAL PHYSICS

On Isobars of the Nucleon Caused by the Kaon Field in a Fixed-Source Theory

by

W. KRÓLIKOWSKI

Presented by W. RUBINOWICZ on February 16, 1959

We consider in this note isobars of the nucleon caused by the kaon field. The problem is analogous to the question of excitations of the nucleon with respect to the pion field, considered previously by many authors by means of the strong coupling approximation (cf. e. g. [1]). A more general method concerning the last question was described in [2]. In the present paper, the method given in [2] is extended to the interaction of the baryon-particle, represented by a fixed source, with the kaon field. The formalism of the baryon-kaon interaction used is that introduced in [3]. This formalism enables us to describe all baryons (N, Λ^0, Σ, Ξ) as states of one baryon-particle (B), just like the isospin formalism enables us to treat, e. g. proton (p) and neutron (n) as isospin states of one nucleon-particle (N). We take into account the Hamiltonian

(1) $$H = H^B + H^\pi + H^K + H^{B\pi} + H^{BK},$$

where

(2) $$H^K = \int d_3x[\pi_a^* \pi_a + \varphi_a^*(-\Delta + \mu_K^2)\varphi_a] \quad \text{and}$$

(3) $$H^{BK} = \frac{f_K}{\mu_K}\sigma_i \xi_a \int d_3x \varrho_K(|\vec{x}|)\partial_i\big(C\varphi(\vec{x})\big)_a + \text{h. c.} + \lambda\left[\int d_3x \varrho_K(|\vec{x}|)\,\xi_a\big(C\varphi(\vec{x})\big)_a + \text{h. c.}\right]^2.$$

Here,

$$\varphi_a(\vec{x}) = K_a(\vec{x}) = \begin{cases} K^+(\vec{x}) & \text{for}\, a=1\,, \\ K^0(\vec{x}) & \text{for}\, a=2\,, \end{cases} \quad \text{and}$$

$$\varphi_a^*(\vec{x}) = \bar{K}_a(\vec{x}) = \begin{cases} K^-(\vec{x}) & \text{for}\, a=1\,, \\ \bar{K}^0(\vec{x}) & \text{for}\, a=2\,. \end{cases}$$

For explanation of the notation used compare [3].

Using the orthonormal expansions

$$\varphi_\alpha(\vec{x}) = \sum_{\lambda l m} q_\alpha(\lambda l m)\, u_{\lambda l}(r)\, Y_{lm}(\vartheta, \varphi) \tag{4}$$

and

$$\pi_\alpha(\vec{x}) = \sum_{\lambda l m} p_\alpha(\lambda l m)\, u^*_{\lambda l}(r)\, Y^*_{lm}(\vartheta, \varphi)\,, \tag{5}$$

where

$$\int\limits_0^\infty r^2 dr\, u^*_{\lambda l} u_{\lambda' l'} = \delta_{\lambda\lambda'}\delta_{ll'}\,, \quad \int\limits_0^{2\pi}\int\limits_0^{2\pi} \sin\vartheta d\vartheta d\varphi\, Y^*_{lm} Y_{l'm'} = \delta_{ll'}\delta_{mm'}\,, \tag{6}$$

we separate from $H^K + H^{BK}$ all kaon degrees of freedom with orbital angular momentum $l \neq 0, 1$. We introduce the notation

$$q_\alpha(\lambda) = q_\alpha(\lambda 00)\,, \qquad q_{\alpha i}(\lambda) = \begin{cases} \dfrac{1}{\sqrt{2}}[q_\alpha(\lambda 11) - q_\alpha(\lambda 1 -1)] & i = 1\,, \\ \dfrac{i}{\sqrt{2}}[q_\alpha(\lambda 11) + q_\alpha(\lambda 1 -1)] & i = 2\,, \\ q_\alpha(\lambda 10) & i = 3\,. \end{cases} \tag{7}$$

$$\omega_l^2(\lambda, \lambda') = \int\limits_0^\infty r^2 dr\, u^*_{\lambda l}(-\Delta + \mu_k^2)\, u_{\lambda' l}\,, \qquad \omega_l^2(\lambda) = \omega_l^2(\lambda, \lambda)\,, \tag{8}$$

$$\varrho_l(\lambda) = \int\limits_0^\infty r^2 dr\, \varrho_k u_{\lambda l}\,, \qquad \varrho_l'(\lambda) = \int\limits_0^\infty r^2 dr\, \frac{d\varrho_k}{dr} u_{\lambda l}\,. \tag{9}$$

Then, we obtain

(10)

$$H^{BK} = 4\pi\lambda \sum_{\lambda\lambda'} \varrho_0^*(\lambda)\, \varrho_0(\lambda')\, q_\alpha^*(\lambda)\, q_\alpha(\lambda') - \sqrt{\frac{4\pi}{3}}\, \frac{f_K}{\mu_K}\, \sigma_i(\xi C)_\alpha \sum_\lambda \varrho_1'(\lambda)\, q_{\alpha i}(\lambda) + \text{h. c.}$$

and

$$\begin{aligned} H^K = & \sum_\lambda [p_\alpha^*(\lambda)\, p_\alpha(\lambda) + \omega_0^2(\lambda)\, q_\alpha^*(\lambda)\, q_\alpha(\lambda)] + \sum_{\lambda \neq \lambda'}\sum \omega_0^2(\lambda, \lambda')\, q_\alpha^*(\lambda)\, q_\alpha(\lambda') \\ & + \sum_\lambda [p_{\alpha i}^*(\lambda)\, p_{\alpha i}(\lambda) + \omega_1^2(\lambda)\, q_{\alpha i}^*(\lambda)\, q_{\alpha i}(\lambda)] + \sum_{\lambda \neq \lambda'}\sum \omega_1^2(\lambda, \lambda')\, q_{\alpha i}^*(\lambda)\, q_{\alpha i}(\lambda') \\ & + \sum_{l \neq 0,1} H_l^K\,. \end{aligned} \tag{11}$$

Now, we assume that the complete orthonormal system of radial functions $u_{\lambda l}(r)$ contains the actual radial functions $u_{\lambda_0 l}(r)$ of the kaon cloud surrounding the core of the baryon. Using the abbreviations

$$q_\alpha = q_\alpha(\lambda_0)\,, \quad q_{\alpha i} = q_{\alpha i}(\lambda_0)\,, \quad \omega_l^2 = \omega_l^2(\lambda_0)\,, \quad \varrho_l = \varrho_l(\lambda_0)\,, \quad \varrho_l' = \varrho_l'(\lambda_0) \tag{12}$$

we get

$$H^K + H^{BK} = \overset{0}{H}_K + \overset{1}{H}_K\,, \tag{13}$$

where

$$\begin{aligned}\overset{0}{H}_K = {} & p_a^* p_a + (\omega_0^2 + 4\pi\lambda|\varrho_0|^2)\, q_a^* q_a \\ & + p_{ai}^* p_{ai} + \omega_1^2 q_{ai}^* q_{ai} - \sqrt{\frac{4\pi}{3}}\, \frac{f_K}{\mu_K}\, \varrho_1' \sigma_i (\xi C)_a q_{ai} + \text{h. c.} \\ & + \sum_{\lambda\neq\lambda_0} [p_a^*(\lambda)\, p_a(\lambda) + \omega_0^2(\lambda)\, q_a^*(\lambda)\, q_a(\lambda)] \\ & + \sum_{\lambda\neq\lambda_0} [p_{ai}^*(\lambda)\, p_{ai}(x) + \omega_1^2(\lambda)\, q_{ai}^*(\lambda)\, q_{ai}(\lambda)] + \sum_{l\neq 0,1} H_l^K\,.\end{aligned} \tag{14}$$

and

$$\begin{aligned}\overset{1}{H}_K = {} & \sum_{\lambda\neq\lambda_0} [\omega_0^2(\lambda_0, \lambda) + 4\pi\lambda\varrho_0^*(\lambda_0)\, \varrho_0(\lambda)]\, q_a^* q_a(\lambda) + \text{h. c.} \\ & + \sum_{\lambda\neq\lambda_0} \left[\omega_1^2(\lambda_0, \lambda)\, q_{ai}^* - \sqrt{\frac{4\pi}{3}}\, \frac{f_K}{\mu_K}\, \varrho_1' \sigma_i (\xi C)_a\right] q_{ai}(\lambda) + \text{h. c.} \\ & + \sum_{\substack{\lambda\neq\lambda' \\ (\lambda\neq\lambda_0\neq\lambda')}}\sum [\omega_0^2(\lambda, \lambda') + 4\pi\lambda\varrho_0^*(\lambda)\, \varrho_0(\lambda')]\, q_a^*(\lambda)\, q_a(\lambda') \\ & + \sum_{\substack{\lambda\neq\lambda' \\ (\lambda\neq\lambda_0\neq\lambda')}}\sum \omega_1^2(\lambda, \lambda')\, q_{ai}^*(\lambda)\, q_{ai}(\lambda')\,.\end{aligned} \tag{15}$$

We shall identify approximately the operator

$$H_K^I = p_a^* p_a + \overline{\omega}_0^2 q_a^* q_a + p_{ai}^* p_{ai} + \omega_1^2 q_{ai}^* q_{ai} - \bar{g}_1 \sigma_i (\xi C)_a q_{ai} + \text{h. c.}, \tag{16}$$

where

$$\overline{\omega}_0^2 = \omega_0^2 + 4\pi\lambda|\varrho_0|^2, \qquad \bar{g}_1 = \sqrt{\frac{4\pi}{3}}\, \frac{f_K}{\mu_K}\, \varrho_1'\,, \tag{17}$$

with a part of the Hamiltonian of the physical baryon, due to the interaction of the bare baryon with the kaon field. In other words, H_K^I is here identified with the Hamiltonian of the kaon cloud interacting with the core of the baryon-particle. The physical baryon has, in this approximation, the following internal degrees of freedom: 1) core degrees of freedom described by σ_i, τ_f, ξ_a, ξ_a^*, 2) pion cloud degrees of freedom introduced in [2] and 3) kaon cloud degrees of freedom given by the canonical co-ordinates q_a, q_a^*, q_{ai}, q_{ai}^* and conjugate momenta p_a, p_a^*, p_{ai}, p_{ai}^*.

The part of the Hamiltonian H_K^I containing the S-wave variables q_a, q_a^*, p_a, p_a^* can be diagonalized immediately. In the P-wave part of H_K^I

we can change the complex variables into real ones using the following formulae

$$\tag{18} \begin{aligned} q_{\alpha i} &= \frac{1}{\sqrt{2}}(q_{\alpha i1} + i q_{\alpha i2}), \qquad p_{\alpha i} = \frac{1}{\sqrt{2}}(p_{\alpha i1} - i p_{\alpha i2}), \\ q^*_{\alpha i} &= \frac{1}{\sqrt{2}}(q_{\alpha i1} - i q_{\alpha i2}), \qquad p^*_{\alpha i} = \frac{1}{\sqrt{2}}(p_{\alpha i1} + i p_{\alpha i2}) \end{aligned}$$

and

$$\tag{19} \begin{aligned} (\xi C)_\alpha &\equiv \epsilon_\alpha = \tfrac{1}{2}(\epsilon_{\alpha 1} - i\epsilon_{\alpha 2}), \\ (\xi C)^*_\alpha &\equiv \epsilon^*_\alpha = \tfrac{1}{2}(\epsilon_{\alpha 1} + i\epsilon_{\alpha 2}). \end{aligned}$$

Let us note that, from the commutation relations (cf. [3])

$$\tag{20} \{\epsilon_\alpha, \epsilon^*_\beta\} = \delta_{\alpha\beta}, \ \{\epsilon_\alpha, \epsilon_\beta\} = 0 \qquad (\alpha, \beta = 1, 2),$$

it follows that

$$\tag{21} \{\epsilon_{\alpha\varrho}, \epsilon_{\beta\sigma}\} = 2\delta_{\alpha\beta}\delta_{\varrho\sigma} \qquad (\alpha, \beta = 1, 2, \ \varrho, \sigma = 1, 2).$$

Then, assuming the constant $\bar{g}_1$ to be real, we get

$$\tag{22} H^I_K = \sum_\alpha (N_\alpha + \overline{N}_\alpha + 1)\overline{\omega}_0 + \frac{1}{2}(p^2_{\alpha i\varrho} + \omega_1^2 q^2_{\alpha i\varrho}) - \frac{\bar{g}_1}{\sqrt{2}}\sigma_i \epsilon_{\alpha\varrho} q_{\alpha i\varrho},$$

where $N_\alpha = 0, 1, 2, \ldots$ and $\overline{N}_\alpha = 0, 1, 2, \ldots$ have the meaning of the numbers of S-wave K_α and $\bar{K}_\alpha$ mesons contained in the kaon cloud.

Let us observe that using the formulae

$$\begin{aligned} K_\alpha &= \frac{1}{\sqrt{2}}(K_{\alpha 1} + iK_{\alpha 2}) & & K_{\alpha 1} = \frac{1}{\sqrt{2}}(K_\alpha + K^*_\alpha) \\ & & \text{or} & \\ K^*_\alpha &= \frac{1}{\sqrt{2}}(K_{\alpha 1} - iK_{\alpha 2}) & & K_{\alpha 2} = \frac{1}{i\sqrt{2}}(K_\alpha - K^*_\alpha) \end{aligned}$$

and (19) we can rewrite the strong baryon-kaon interaction given in [3],

$$\mathcal{H}^{BK} = ig_K \overline{B}\gamma_5 (\xi C)_\alpha B K_\alpha + \text{h. c.},$$

in the form

$$\mathcal{H}^{BK} = i\frac{g_K}{\sqrt{2}}\overline{B}\gamma_5 \epsilon_{\alpha\varrho} B K_{\alpha\varrho} = i\frac{g_K}{\sqrt{2}}\overline{B}\gamma_5 \delta_\mu B \varkappa_\mu,$$

where

$$\varkappa_1 = K_{11}, \qquad \varkappa_2 = K_{12}, \qquad \varkappa_3 = K_{21}, \qquad \varkappa_4 = K_{22}$$

and

$$\delta_1 = \epsilon_{11}, \qquad \delta_2 = \epsilon_{12}, \qquad \delta_3 = \epsilon_{21}, \qquad \delta_4 = \epsilon_{22}.$$

It follows from (21) that

$$\{\delta_\mu, \delta_\nu\} = 2\delta_\mu \qquad (\mu, \nu = 1, 2, 3, 4).$$

We note that $K_{21} = K_1^0$ and $K_{22} = K_2^0$ are the neutral K mesons introduced by Gell-Mann and Pais [4].

If we assume, according to [3],

$$H^{B\pi} = \frac{f_\pi}{\mu_\pi}\sigma_i(\tau_j + \xi_\alpha^* \tau_{j\alpha\beta}\xi_\beta)\int d_3x\varrho_\pi(|\vec{x}|)\partial_i\varphi_j(\vec{x}) \tag{23}$$

and use the results presented in [2], we shall approximately identify the total Hamiltonian of the physical baryon with the operator

$$\begin{aligned} H^I = (\text{core mass term}) + \frac{1}{2}(p_{ij}^2 + \omega^2 q_{ij}^2) - \bar{g}\sigma_i(\tau_j + \xi_\alpha^* \tau_{j\alpha\beta}\xi_\beta)q_{ij} \\ + \sum_\alpha (N_\alpha + \overline{N}_\alpha + 1)\overline{\omega}_0 + \frac{1}{2}(p_{\alpha i\varrho}^2 + \omega_1^2 q_{\alpha i\varrho}^2) - \frac{\bar{g}_1}{\sqrt{2}}\sigma_i\epsilon_{\alpha\varrho}q_{\alpha i\varrho}\,. \end{aligned} \tag{24}$$

Eigenstates of the Hamiltonian (24) are, therefore, identified in this theory with the states of the physical baryon. Excited eigenstates of (24) may be called isobars of the baryon (in particular isobars of the nucleon). The eigenstates and eigenvalues of the Hamiltonian (24) can be numerated among others by the eigenvalues of operators $\vec{J}^{I2}$, J_3^I, $\vec{T}^{I2}$, T_3^I, S^I, where

$$\vec{J}^I = \tfrac{1}{2}\vec{\sigma} + (\overrightarrow{q_{ik}p_{jk} - q_{jk}p_{ik}}) + (\overrightarrow{q_{\alpha i\varrho}p_{\alpha j\varrho} - q_{\alpha j\varrho}p_{\alpha i\varrho}})\,, \tag{25}$$

$$\vec{T}^I = \frac{1}{2}(\vec{\tau} + \xi_\alpha^*\vec{\tau}_{\alpha\beta}\xi_\beta) + (\overrightarrow{q_{ki}p_{kj} - q_{kj}p_{ki}}) + \frac{1}{2i}(p_\alpha\vec{\tau}_{\alpha\beta}q_\beta + p_{\alpha i}\vec{\tau}_{\alpha\beta}q_{\beta i}) + \text{h. c.}\,, \tag{26}$$

and

$$\begin{aligned} S^I &\doteq \xi_\alpha^*\xi_\alpha - \sum_\alpha (N_\alpha - \overline{N}_\alpha) - \sum_{\alpha i}(N_{\alpha i} - \overline{N}_{\alpha i}) \\ &= 1 - \sum_\alpha (N_\alpha - \overline{N}_\alpha) - \frac{1}{2i}\epsilon_{\alpha 1}\epsilon_{\alpha 2} - (q_{\alpha i 1}p_{\alpha i 2} - q_{\alpha i 2}p_{\alpha i 1}) \end{aligned} \tag{27}$$

are the total spin, the total isospin and the total strangeness of the physical baryon, respectively. Here, $\sum_\alpha n_\alpha = \xi_\alpha^*\xi_\alpha$ is the strangeness of the core, N_α and $\overline{N}_\alpha$ are, as in formula (22), the numbers of S-wave K_α and $\overline{K}_\alpha$ mesons present in the kaon cloud, $\sum_i N_{\alpha i}$ and $\sum_i \overline{N}_{\alpha i}$ — the operators of the numbers of P-wave K_α and $\overline{K}_\alpha$ mesons in this cloud (these operators and also $n_\alpha = \xi_\alpha^*\xi_\alpha$ do not commute with H^I). The total charge of the physical baryon is given by

$$Q^I = e\left(T^I + \frac{1}{2} - \frac{S^I}{2}\right), \qquad e = |e|\,. \tag{28}$$

The exact solving of the eigenvalue problem for the Hamiltonian (24) seems to be a formidable task. Some approximation methods, for instance similar to the strong coupling approach (cf. [1]), may be tried in order to solve this problem (cf. [2]). We confine ourselves in this note to the above qualitative results showing (see (24)) the possibility of high energy excitations of the nucleon caused by the baryon-kaon interaction (in a rough estimate, we have $\bar{\omega}_0 \sim \mu_K$ or $10\,\mu_K$). These excited states should appear besides low and moderately high energy excitations connected with baryon-pion interaction (cf. e. g. [1] and [2]; the lowest excitation of this kind corresponds to the well known $\frac{3}{2}, \frac{3}{2}$ resonance [5]). The considerations of the present paper lead to the possibility of excitations connected with both the S-wave and P-wave kaons in the cloud of the nucleon. The possibility of the $P_{3/2}$-wave kaon excitations can also be supported by an argument based on the effective range approach of the Chew-Low type (cf. [6]).

INSTITUTE OF THEORETICAL PHYSICS, WARSAW UNIVERSITY
(INSTYTUT FIZYKI TEORETYCZNEJ UNIWERSYTETU WARSZAWSKIEGO)
INSTITUTE FOR NUCLEAR RESEARCH, POLISH ACADEMY OF SCIENCES
(INSTYTUT BADAŃ JĄDROWYCH PAN)

REFERENCES

[1] W. Pauli and S. M. Dancoff, Phys. Rev., **62** (1942), 85.
[2] W. Królikowski, Bull. Acad. Polon. Sci., Série des sci. math., astr. et phys., **6** (1958), 413.
[3] — Nuclear Physics, **8** (1958), 461.
[4] M. Gell-Mann and A. Pais, Phys. Rev., **97** (1955), 1387.
[5] Proceedings of Geneva Conference, CERN (1958).
[6] W. Królikowski (in this issue).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

THEORETICAL PHYSICS

A Six-Dimensional Interpretation of Nuclear Forces

by

J. RAYSKI

Presented by L. INFELD on February 25, 1959

Let us start with the traditional equation for the nucleon coupled to the pion field

$$\left(\sum_{k=1}^{4}\gamma_k\partial_k+ig\gamma_5\vec{\tau}\vec{\Phi}\right)\psi=0\,. \tag{1}$$

The mass term has been dropped in (1) for the sake of simplicity on the ground that the nucleonic mass is probably a consequence of an interaction so that there is no need for the mass term to appear in a fundamental equation. The nucleonic field quantity has two indices $\psi_{\alpha\varrho}$, where $\alpha=1,\dots 4$ is a spinor index with respect to the Lorentz group and $\varrho=1,2$ is the index of isospin.

In order to find an interpretation of isobaric (isotopic) degrees of freedom we shall rewrite (1) in a different form taking advantage of an eight-dimensional notation

$$\begin{aligned}&\psi_{11}=\Psi_1, \quad \psi_{21}=\Psi_2, \quad \psi_{31}=\Psi_3, \quad \psi_{41}=\Psi_4,\\ &\psi_{12}=\Psi_5, \quad \psi_{22}=\Psi_6, \quad \psi_{32}=\Psi_7, \quad \psi_{42}=\Psi_8.\end{aligned} \tag{2}$$

In this notation Eq. (1) appears in an interesting form

$$\left(\sum_{k=1}^{4}\Gamma_k\partial_k+ig\sum_{\mu=5}^{7}\Gamma_\mu\varphi^\mu\right)\Psi=0\,, \tag{3}$$

where

$$\varphi^5=\Phi^1, \quad \varphi^6=\Phi^2, \quad \varphi^7=\Phi^3. \tag{4}$$

The matrices Γ_μ (Greek indices run from 1 to 7, Latin from 1 to 4) are Dirac matrices with eight rows and columns

$$\Gamma_k = \begin{pmatrix} \gamma_k & \\ & \gamma_k \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} & \gamma_5 \\ \gamma_5 & \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} & -i\gamma_5 \\ i\gamma_5 & \end{pmatrix} \tag{5}$$

and

$$\Gamma_7 = \begin{pmatrix} \gamma_5 & \\ & -\gamma_5 \end{pmatrix} = -i\Gamma_1\Gamma_2 \ldots \Gamma_6 . \tag{6}$$

They satisfy the usual relations

$$\Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\delta_{\mu\nu} . \tag{7}$$

The new form (3) of the Dirac equation satisfied by the nucleon possesses conspicuous features revealing the deeper sense of the isospin and its connection with the electric charge. It also throws new light upon the question, why pions are *ps*-scalar.

First of all let us recall that eight-component spinors are geometrical objects fitting most naturally into a six- or a seven-dimensional space [1]. Thus, the appearance of the seven matrices Γ_μ in (3) and the simplicity and elegance of the interaction term appearing in (3) in contradistinction to the clumsy form of (1) suggest that the physical space is not four- but six- or seven-dimensional. We shall limit ourselves to the six-dimensional viewpoint.

Assume that the Minkowski space $(1, \ldots, 4)$ is embedded in a six-dimensional space $(1, \ldots, 6)$. The matrices Γ_5 and Γ_6 are connected with the fifth and sixth dimensions in a similar sense as $\Gamma_1, \ldots, \Gamma_4$ with the usual four dimensions, whereas Γ_7 defined by (6) plays the same role in the six-dimensional world as γ_5 does in the four-dimensional world.

The pion field is a vector and the nucleonic field is a spinor in a three-dimensional "space" formed by two genuine dimensions 5 and 6 and by an imitation of a dimension 7 brought about by the algebraic fact that, if we have six matrices satisfying (7), then there must also exist a seventh matrix of the same type being a product of the remaining six. Since not only Γ_7 but also Γ_5 and Γ_6 can be regarded as products of the remaining six matrices, neither of them is privileged, which accounts for the appearance of an imitation of a three-dimensional space (5, 6, 7) in spite of the fact that the physical space is actually only six-dimensional.

In view of this situation it is understandable that some physical phenomena (nuclear interactions) can take advantage of the algebraic symmetry brought about by the existence of three subsidiary matrices $\Gamma_5, \Gamma_6, \Gamma_7$, but some other phenomena (electromagnetic and weak interactions) can be sensitive to the fact that the space is actually six-dimensional and that the seventh dimension is not genuine. In this way we can understand qualitatively the most enigmatic feature of "iso-

space": why not all but only some laws of physics possess a symmetry with respect to rotation in "isospace".

The charged meson field is usually described in a complex form. In our notation we have

$$\varphi = \varphi^5 + i\varphi^6, \qquad \varphi^* = \varphi^5 - i\varphi^6. \tag{8}$$

Herefrom it is seen that gauge transformations acquire a geometrical meaning of rotations in the subspace (5, 6) and that the charge conjugation means an inversion (of the axis 6) in this subspace *). Thus, the subspace (5, 6) should be interpreted as an electric space in the following sense: a rotation (spin) in the electric subspace manifests itself experimentally as the electric charge.

In this way we get the following picture of the physical space: our world is not a four-dimensional manifold of points but a four-dimensional manifold of two-dimensional electric surfaces (or surface elements) orthogonal to the usual four dimensions. The electric charge is a manifestation of an intrinsic rotation in the subspace (5, 6), but the appearance of a matrix Γ_7 on equal footing with Γ_5 and Γ_6 offers a possibility of a higher symmetry in a three-dimensional "space" (5, 6, 7). This explains the isospin and the distinguished role of its "third" component.

Moreover, we can understand why pions must be *ps*-scalar. First of all let us recall that the *ps*-scalar or scalar character of charged pions is purely a matter of convention since we can reverse arbitrarily the sign of φ^5 and φ^6 by a gauge transformation: a rotation in the subspace (5, 6) through the angle π. Therefore, we have to investigate only the properties of neutral pions described by φ^7. Since the "axis 7" is not genuine but is induced by the existence of a matrix Γ_7 being a product of the remaining six matrices, an inversion of an axis $k=1, \ldots, 4$ changes automatically the sign of expressions involving Γ_7 which means an inversion of the "axis 7" and consequently changes the sign of the "vector component" φ^7. In this way it is seen that neutral pions are *ps*-scalar with respect to inversions in the Minkowski subspace.

We can also argue as follows: since (from a six-dimensional viewpoint) the interaction involved in (3) constitutes the simplest and the most natural possibility (at least in the case of a meson field forming a charged triplet), and a transition back to the traditional notation (1) leads automatically to the appearance of a γ_5 in the interaction term, we see that a *ps*-scalar interaction with a *ps*-scalar meson field means the simplest possibility, by far more natural than a scalar coupling with a scalar meson field.

The six-dimensional viewpoint can be extended to the case of strong interactions involving hyperons and heavy mesons. Moreover, the con-

---

*) The same interpretation applies also to fermions in a Majorana representation.

ception of a six-dimensional space enables a geometrical interpretation of the electromagnetic field (compare, e. g. [2] or [3]).

The problem of the electromagnetic interaction in terms of a six-dimensional Riemannian manifold, as well as the problem of heavy mesons and hyperons, strangeness, etc., will be dealt with in detail in a separate paper in Acta Physica Polonica.

INSTITUTE OF PHYSICS, JAGELLONIAN UNIVERSITY, CRACOW
(INSTYTUT FIZYKI UNIWERSYTETU JAGIELLOŃSKIEGO, KRAKÓW)
INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI PAN)

REFERENCES

[1] E. Cartan, *Leçons sur la théorie des spineurs*, Hermann, Paris, 1938.
[2] J. Podolansky, Proc. Royal Soc., A **201** (1950), 234.
[3] J. Rayski, Acta Phys. Polon., **17** (1958), 187.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sci. math., astr.
et phys.—Vol. VII, No. 4, 1959

EXPERIMENTAL PHYSICS

Electron Bombardment Induced Conductivity in Ge

by

A. ZARĘBA

Presented by A. SOŁTAN on February 3, 1959

Introduction

The conductivity induced in Ge by impact ionization from electron bombardment of a monocrystalline sample was investigated.

The conductivity thus arising in semiconductors is termed β-conductivity by analogy to photoconductivity, independently of the energies of the electrons used. Similarly the notion of β-current is introduced. In addition to impact ionization, electron bombardment of semiconductors may introduce defects in the crystal lattice; this occurs when the energy exceeds a threshold value characteristic of the semiconductor. In the case of Ge, the threshold value is 510 keV [1]. β-conductivity was observed i. a. in the bombardment of homogeneous CdS monocrystals [2] and n-p junctions in Ge [3]-[6].

In the course of the present investigation it was found that an homogeneous n-type Ge monocrystal exhibits β-conductivity. The volume effect, as distinct from the barrier effect, makes it possible to study more closely the impact ionization introduced by electron bombardment. It is the aim of the present investigation to determine the average effective energy ε needed to generate an electron-hole pair, this energy being the measure of the yield in the effect of β-conductivity.

Method of measurement

The measurements were carried out in an EM3 electron microscope of Soviet production, especially adapted for the experiment. Fig. 1 shows a block diagram of the device. The electron gun, electronic-optical system and sample are situated within the vacuum chamber E. The electron beam from the cathode K is accelerated by the voltage supply L and deviated by the plates P.

The pulses to the plates P are provided by the rectangular pulse generator G. Pulse bombardment of the sample was chosen in order to eliminate from the measurements temperature variations of the resistivity occurring during bombardment, and for selective amplification of the very small signals appearing in the measurements.

The β-current Δi arising in the sample A gives rise to the voltage

$$\Delta U' = -R\Delta i \tag{1}$$

across the load resistance $R = R_1 = R_2$. In the conditions of experiment, the current of primary electrons I may be neglected with respect to Δi.

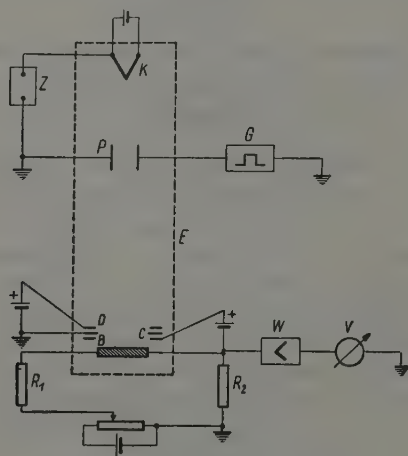

Fig. 1. Block diagram of device

The voltage $\Delta U'$ is amplified by the selective amplifier W and is measured by the voltmeter V. The diaphragm B is situated above the sample for capturing the secondary electrons leaving the sample. Diaphragm C protects diaphragm B from the bombarding electrons. Diaphragm D captures the secondary electrons from C so that they cannot reach diaphragm B.

The β-current is given by the following expression:

$$\Delta i = [p_0+\Delta p(x)]\mu_h e[E-\Delta E(x)]+[n_0+\Delta n(x)]\mu_e e[E-\Delta E(x)] - [p_0\mu_h+n_0\mu_e]eE \tag{2}$$

wherein p_0, and n_0 denote the concentration per unit length of holes and electrons within the sample in the absence of β-bombardment,

$\Delta p(x)$, $\Delta n(x)$ — the respective additional concentrations per unit length produced by β-bombardment,

E — the homogeneous field within the sample, and

$\Delta E(x)$ — the additional field arising therein from β-bombardment.

In the foregoing expression, appropriate values of $\Delta E(x)$ guarantee continuity of the β-current.

Eq. (2) is now applied to the n-type Ge. Thus, $p_0 \ll n_0$.

Moreover, it is assumed that $\Delta p \ll p_0$, $\Delta n \ll n_0$, $\Delta E \ll E$ yielding

$$\Delta i = [\Delta p(x)\mu_h + \Delta n(x)\mu_e]eE - n_0\mu_e e\Delta E . \tag{3}$$

The β-current may also be expressed as follows:

$$\Delta i = \frac{1}{l}\int_0^l \Delta i\, dx . \tag{4}$$

Integration of Eq. (3), with the following assumptions:

$N_0 = n_0 l$, wherein l denotes the length of the sample,

$\Delta P = \int_0^l \Delta p(x)\, dx$,

$\Delta N = \int_0^l \Delta n(x)\, dx$,

$U = El$, wherein U is the voltage on the sample,

and

$\Delta U = \int_0^l \Delta E(x)\, dx$, wherein ΔU is the voltage variation on the sample,

yields

$$\Delta i = \frac{1}{l^2}\mu_h e\Delta P U + \frac{1}{l^2}\mu_e e\Delta N U - \frac{1}{l^2}\mu_e e N_0 \Delta U .$$

Zero charge of the sample as a whole, $\Delta N = \Delta P$, yields

$$\Delta i = \frac{1}{l^2}(\mu_h + \mu_e) e\Delta P U - \frac{1}{l^2}\mu_e e N_0 \Delta U . \tag{5}$$

By Eq. (1),

$$-\Delta U' = \frac{R}{l^2}(\mu_h + \mu_e) e\Delta P U - \frac{R}{l^2}\mu_e e N_0 \Delta U .$$

By $R = R_1 = R_2$, we have $\Delta U = -2\Delta U'$, with ΔU and $\Delta U'$ denoting the voltage variations on the sample and on the load resistance R, respectively; hence

$$-\Delta U' = \frac{\frac{R}{l^2}(\mu_h + \mu_e) e\Delta P U}{1 + 2\frac{R}{r}} ,$$

wherein r denotes the resistance of the sample.

Since $|\Delta U'|$ is the only quantity measured, we have:

$$\Delta U' = \frac{1}{1 + 2\frac{R}{r}} \frac{R}{l^2}(\mu_h + \mu_e) e\Delta P U . \tag{6}$$

The quantity $p(x)$ is obtained from the equation of continuity for the additional holes:

$$\frac{\partial}{\partial t}\Delta p = \frac{g}{l} - \frac{1}{\tau}\Delta p - div j\,, \tag{7}$$

where τ is the effective lifetime of the additional carriers, as resulting from both bulk and surface recombination, g is the number of electron-hole pairs generated within the sample,

$$g = \frac{V_0}{\varepsilon}\nu\,, \tag{8}$$

with V_0 denoting the energy of the electrons used for bombardment, ν — the number of electrons incident on the sample per second, and ε — the energy of generation per electron-hole pair.

Eq. (7) at equilibrium, which is

$$\frac{\partial}{\partial t}\Delta p = 0\,,$$

leads to

$$\frac{g}{l} - \frac{1}{\tau}\Delta p - \mu_h E\frac{\partial}{\partial x}\Delta p + D\frac{\partial^2}{\partial x^2}\Delta p = 0\,. \tag{9}$$

Fig. 2 shows the geometry of the sample. The field is applied along the x-axis.

Substituting $\mu_h E\tau = \Lambda$, $D\tau = L^2$ and assuming $\Lambda^2 > 4L^2$ leads to the following solution of Eq. (9):

$$\Delta p(x) = \frac{g}{l}\tau - Ce^{-x/\Lambda}.$$

The constant C is obtained from the condition $\Delta p(0) = 0$ holding in the case of exclusion [7], [8].

Finally,

$$\Delta p(x) = \frac{g}{l}\tau(1-e^{-x/\Lambda})\,, \tag{10}$$

$$\Delta P = g\tau\left[1 + \frac{\Lambda}{l}(e^{-1/\frac{\Lambda}{l}} - 1)\right]. \tag{11}$$

Substituting (11) into Eq. (6) yields

$$\Delta U' = \frac{1}{1+2\dfrac{R}{r}}\frac{R}{l^2}(\mu_h+\mu_e)I\frac{V_0}{\varepsilon}\tau\left[1+\frac{\Lambda}{l}(e^{-1/\frac{\Lambda}{l}}-1)\right]U, \tag{12}$$

wherein $I = \nu e$ is the intensity of the electron beam incident on the sample.

Eq. (12), which serves for determining ε, contains certain quantities obtained from measurements. The latter, generally, present no difficulties.

In measuring I the effect of secondary emission of electrons from the surface of the semiconductor was eliminated. However, the effective time τ in the equation gives rise to certain difficulties of an essential nature. Some authors use the lifetime τ determined by another method. It seems that this may be a source of error, since it is reasonable that electron bombardment should involve processes on the surface of the sample modifying the surface recombination and, hence, the effective lifetime τ. For example, in sample No. 31, electron bombardment was observed to reduce τ to one half its value as compared with the experimental value of τ obtained from the photocurrent.

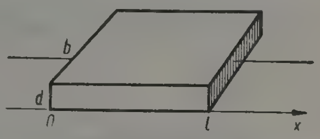

Fig. 2. Geometry of the sample

Hence, it is imperative that τ be determined directly from the effect of β-conductivity, for one and the same sample, and in the conditions prevailing when measuring ε. This implies that two parameters, namely ε and τ, are determined simultaneously from Eq. (12), which may be done if (12) is considered to express $\Delta U'$ as a function of U. By varying the voltage U on the sample and measuring $\Delta U'$ at constant values of the remaining parameters, the experimental dependence $\Delta U'(U)$ was obtained.

Eq. (12) may be rewritten as follows:

$$\Delta U' = \alpha[(\gamma U) + (\gamma U)^2 (e^{-1/\gamma U} - 1)] \tag{13}$$

with

$$\alpha = \frac{R}{1 + 2\frac{R}{r}} \frac{\mu_h + \mu_e}{\mu_h} I \frac{V_0}{\varepsilon}, \tag{14}$$

$$\gamma = \frac{\mu_h}{l^2} \tau. \tag{15}$$

α and γ were determined from experimentally found function $\Delta U'(U)$. Hence, by (14) and (15),

$$\varepsilon = \frac{R}{1 + 2\frac{R}{r}} \frac{\mu_h + \mu_e}{\mu_h} I V_0 \frac{1}{\alpha}, \tag{16}$$

$$\tau = \frac{l^2}{\mu_h} \cdot \gamma. \tag{17}$$

Results of measurements

Germanium of type n possessing a resistivity of $\varrho = 4\,\Omega$ cm was investigated. Tin electrodes served for showing the ohm resistivity of the contacts. The dimensions of sample No. 31 for which the results are given were: $1 = 0.9$ mm, $d = 0.12$ mm, $b = 1.4$ mm. Previous to the

measurements, the sample was etched in H_2O_2 to reduce surface recombination. The energy of the electrons bombarding the sample was 22.7 keV. The intensity of the electron beam amounted to $I = 3.4 \times 10^{-9}$ A; noise was negligible.

The points plotted in Fig. 3 represent the experimental function $\Delta U'(U)$ dependence, yielding

$$\alpha = 1.23 \cdot 10^{-2},$$
$$\gamma = 0.40 .$$

The continuous line in Fig. 3 shows $\Delta U'(U)$ as found from (13) with the foregoing values of the coefficients α and γ. The fact that the experimental results are in good agreement with the theoretical curve is an indication that the mechanism proposed together with the approximations used account for the effect in a satisfactory manner.

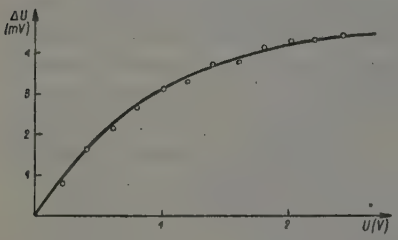

Fig. 3. β-current voltage across load resistance versus voltage on sample

The parameters required, as computed from (16) and (17) are

$$\varepsilon = 2.8 \pm 0.3 \text{ eV},$$
$$\tau = 1.8 \pm 0.2 \ \mu\text{sec} .$$

This value of ε may be compared not only with the results of papers already quoted [3]-[6], but also with those of papers [9], [10] dealing with impact ionization caused by other factors.

The author wishes to thank Professor L. Sosnowski for directing the present investigation and for his valuable discussions, and is indebted to Mr. A. Feltynowski for suggesting the use of the electron microscope in the measurements, and for putting the instrument at the author's disposal.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES
(INSTYTUT FIZYKI PAN)

REFERENCES

[1] J. J. Loferski, P. Rappaport, Phys. Rev., **98** (1955), 1861.

[2] S. M. Ryvkin, B. M. Konovalenko, J. S. Smetannikova, J. techn. phys., **24** (1954), 961.

[3] A. R. Moore, F. Hermyn, Phys. Rev., **81** (1951), 472.

[4] P. Rappaport, Phys. Rev., **93** (1954), 246.

[5] U. B. Soltanov, E. S. Grishin, J. Techn. Phys., **28** (1958), 1394.

[6] V. S. Vavilov, L. S. Smirnov, V. M. Packevitsh, DAN, **112** (1957), 1020.

[7] R. Bray, Phys. Rev., **100** (1955), 1047.

[8] S. Mayburg, Rev. Sc. Instr., **26** (1955), 616.

[9] J. Drankoupil, M. Malkovska, J. Tauc, Tschech. J. Phys., **7** (1957), 57.

[10] K. G. Mc Kay, K. B. Mc Affree, Phys. Rev., **91** (1951), 1079.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

Резюме статей

ТОМ VII 1959 ВЫПУСК 4

А. ШИБЯК, ОБ АСИМПТОТИЧЕСКОМ ПОВЕДЕНИИ РЕШЕНИЙ УРАВНЕНИЯ $\Delta u - \frac{\partial u}{\partial t} + c(x)u = 0$ . . . . стр. 183—186

Через $c(x)$ обозначим ограниченную, неотрицательную функцию, неравную тождественно нулю, определенную в m-мерном эвклидовом пространстве и удовлетворяющую условию Гельдера. Пусть $U(t, x, s, y)$ — фундаментальное решение уравнения $\Delta u - \frac{\partial u}{\partial t} + c(x)u = 0$. При t, стремящемся к бесконечности, функция $U(t, x, s, y)$ стремится к бесконечности квази равномерно относительно x и y. Эта сходимость показательного типа.

К. ГЕМБА и З. СЕМАДЕНИ, О ВЛОЖЕНИИ ОДНОГО ПРОСТРАНСТВА НЕПРЕРЫВНЫХ ФУНКЦИЙ В ДРУГОЕ . . . . стр. 187—189

Пусть Ω и Ω_0 — компактные пространства Хаусдорффа. $C(\Omega)$ и $C(\Omega_0)$ — банаховы структуры непрерывных действительных функций, определенных на Ω и Ω_0, а $e(t)$ и $e_0(t)$ — функции тождественно равные 1 соответственно на Ω и Ω_0.

Предположим, что T взаимно однозначная, линейная, изометрическая и изотоническая операция, отображающая $C(\Omega_0)$ на некоторое подпространство X_0 пространства $C(\Omega)$.

Положим:

$$\Omega_1 = \{t \in \Omega : Te_0(t) = e(t)\} \cap \bigcap_{x,y \in X_0} \{T(x \vee y)(t) = \max[Tx(t), Ty(t)]\}.$$

В работе приводятся (без доказательств) следующие теоремы:

1° $\sup\limits_{t \in \Omega_1} |x(t)| = \|x\|$, для любого $x \in X_0$,

2° существует непрерывное отображение Ω_1 на Ω_0,

3° если существует неотрицательная проекция P с нормой равной 1 пространства $C(\Omega)$ на X_0, то $\Omega_0 \underset{\text{top}}{\subset} \Omega$.

А. ГРАНАС, О ПОНЯТИИ ТОПОЛОГИЧЕСКОЙ СТЕПЕНИ ДЛЯ ОДНОГО КЛАССА МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ В БАНАХОВЫХ ПРОСТРАНСТВАХ . . . . стр. 191–194

В работе рассматриваются отображения вида $x-\Phi(x)$, где Φ — вполне непрерывный, многозначный, полунепрерывный сверху оператор, действующий в банаховом пространстве X; значения оператора Φ выпуклы в X. Для таких отображений определяется топологическая степень Лерэ-Шаудера.

С. РОЛЕВИЧ, НЕКОТОРЫЕ ЗАМЕЧАНИЯ О ЛИНЕЙНЫХ ПРОСТРАНСТВАХ МОНТЭЛЯ . . . . стр. 195–197

В настоящей работе показана связь, существующая между аппроксимативной размерностью А. Н. Колмогорова и пространствами типа Шварца. При помощи аппроксимативной размерности приводится характеристика ядерных пространств с абсолютным базисом.

Кроме того, дается теорема о том, что всякое бесконечномерное пространство Монтэля содержит бесконечномерное ядерное пространство с абсолютным базисом.

Л. ВЛОДАРСКИЙ, ОБЩИЕ МЕТОДЫ ОГРАНИЧЕНИЯ ТИПА БОРЕЛЯ . . . . стр. 199–200

Методы Бореля „экспоненциальный тип" B_α ($\alpha > 0$, произвольные), которые определяют обобщенную границу ξ рядов $\{\xi_n\}$ при помощи формулы

$$B_\alpha(t, x) = \alpha e^{-t} \sum_{n+0}^{\infty} \frac{t^{\alpha n}}{\Gamma(\alpha n+1)} \xi_n$$

являются непрерывными, регулярными, способными перемещаться вправо, взаимно-конформными, а также конформными Абелевому методу.

В. ЖЕЛЯЗКО, О НЕКОТОРОМ КЛАССЕ ТОПОЛОГИЧЕСКИХ ТЕЛ . . . . стр. 201–203

В работе приводятся следующие результаты:

Теорема 1. *Пусть R — метрическая, полная алгебра с делением над полем действительных чисел. Если в R существует ограниченная окрестность нуля или метрика является подмультипликативной, то R есть или поле действительных чисел или поле комплексных чисел, или поле кватерионов.*

Теорема 2. *Если R — нормированная алгебра над полем действительных чисел и норма удовлетворяет условию $\|x^{-1}\| = \|x\|^{-1}$, то тогда имеет место утверждение теоремы* 1.

Теорема 3. *Если R — нормированная алгебра над полем действительных чисел и если она удовлетворяет условию $\|xy\| = \|x\|\,\|y\|$ для $x, y \in R$, то тогда имеет место утверждение теоремы* 1.

Теорема 4. *Если R есть m-выпуклая алгебра с делением над полем действительных чисел, то тогда имеет место утверждение теоремы* 1.

В. ПОГОЖЕЛЬСКИЙ, КРАЕВАЯ ЗАДАЧА С КАСАТЕЛЬНЫМИ ПРОИЗВОДНЫМИ ДЛЯ ЭЛЛИПТИЧЕСКОГО УРАВНЕНИЯ . . . . стр. 205–212

Задача состоит в определении функции $u(A)$, которая в каждой точке $A(x_1, \ldots, x_n)$ области Ω в n-мерном эвклидовом пространстве удовлетворяла бы эллиптическому уравнению (1) второго порядка и которая в каждой точке P поверхности S, ограничивающей область Ω, выполняла бы краевое соотношение (3). В этом соотношении du/dT_p обозначает трансверсальную производную неизвестной функции, $u_{s^{(a)}_P}(P)$ — предельные значения производных функций $u(A)$ в направлении касательных $s^{(t)}, \ldots, s^{(q)}$, связанных с каждой точкой P поверхности S. Автор, опираясь на свойства потенциалов, относительно уравнения (1), привел задачу к системе (15) сингулярных интегральных уравнений и доказал существование решения задачи при помощи топологической теоремы Шаудера о неподвижной точке преобразования.

В. ШМЕЛЕВА, АБСОЛЮТНОЕ ИСЧИСЛЕНИЕ ОТРЕЗКОВ И ЕГО МЕТАМАТЕМАТИЧЕСКИЕ ПРИМЕНЕНИЯ . . . . стр. 213–220

В настоящей работе автор дает некоторое исчисление отрезков в элементарной абсолютной геометрии. Это исчисление, в случае эвклидовой геометрии, совпадает с исчислением отрезков, построенным Гильбертом [1], а в случае геометрии Болиаи Лобачевского, является весьма приближенным к гиперболическому исчислению отрезков, приведенному автором [4].

Опираясь на абсолютное исчисление отрезков, автор получает новое доказательство теоремы о репрезентации для элементарной абсолютной геометрии.

М. ФИШ, О НЕОБХОДИМЫХ И ДОСТАТОЧНЫХ УСЛОВИЯХ ДЛЯ УСИЛЕННОГО ЗАКОНА БОЛЬШИХ ЧИСЕЛ, ВЫРАЖЕННЫХ В ТЕРМИНАХ МОМЕНТОВ . . . . стр. 221–225

Теорема 1. *Для последовательности независимых, симметрических, случайных переменных X_i $(i = 1, 2, \ldots)$, удовлетворяющих неравенствам*

$$|X_i| < i,$$

не существуют необходимые и достаточные условия выполнения усиленного закона больших чисел, выраженные в терминах дисперсии.

Теорема 2. *Для произвольной последовательности независимых, случайных, переменных не существуют необходимые и достаточные условия для выполнения усиленного закона больших чисел, выраженные в терминах моментов порядка $1, 2, 3, \ldots, r$ при каком либо конечном r.*

З. ЦЕСЕЛЬСКИЙ, О ФУНКЦИЯХ ХААРА И О БАЗИСЕ ШАУДЕРА В ПРОСТРАНСТВЕ $C_{\langle 0,1\rangle}$ . . . . стр. 227—232

Пусть $C_{\langle 0,1\rangle}$ — множество всех непрерывных функций $x(t)$, заданных на отрезке $\langle 0,1\rangle$, с обычной нормой $\|x\| = \max\limits_{\langle 0,1\rangle} |x(t)|$. Обозначим через $\{\chi_n(t)\}$, $n = 1, 2, \ldots$, систему Хаара ([1], стр. 44), а через $\{\varphi_n(t)\}$, систему определенную следующим образом: $\varphi_0(t) \equiv 1$, $\varphi_n(t) \int\limits_0^t = \chi_n(\tau)\,d\tau$, $0 \leqslant t \leqslant 1$, $n = 1, 2, \ldots$ Пусть

$$S_n(t) = \sum_{k=1}^{n} a_k \chi_k(t), \quad \text{где} \quad a_k = \int_0^1 x(t)\,\chi_k(t)\,dt \quad \text{для} \quad k = 1, 2, \ldots,$$

$$T_n(t) = \sum_{k=0}^{n} b_k \chi_k(t), \quad \text{где} \quad b_0 = x(0),\ b_k = \int_0^1 \chi_k(t)\,dx(t) \quad \text{для} \quad k = 1, 2, \ldots$$

и пусть для $0 \leqslant \delta \leqslant 1$.

$$\omega_1(\delta) = \sup_{\substack{t_3, t_2 \in \langle 0,1\rangle \\ |t_1 - t_2| \leqslant \delta}} |x(t_1) - x(t_2)|, \quad \omega_2(\delta) = \sup_{\substack{t_1, t_2 \in \langle 0,1\rangle \\ |t_1 - t_2| \leqslant \delta}} \left| x(t_1) + x(t_2) - 2x\left(\frac{t_1 + t_2}{2}\right) \right|.$$

Скажем, что $\omega(t)$ удовлетворяет условию $(*)$, если $\int\limits_0^\delta \omega(t)/t\,dt \leqslant K\omega(\delta)$ для $0 \leqslant \delta \leqslant 1$ и, что $\omega(t)$ удовлетворяет условиям $(**)$, если: $\omega(t) \geqslant 0$ не возрастает с убыванием t в промежутке $\langle 0,1\rangle$ и $\omega(2t) \leqslant M\omega(t)$ для $0 \leqslant 2t \leqslant 1$.

Теорема 1. *Если* $\omega_1(\delta)$ *и* $\omega_2(\delta)$ *удовлетворяют условию* $(*)$, *тогда* $\|x - S_n\| \leqslant \dfrac{2k}{\log 2}\,\omega_1\!\left(\dfrac{1}{n}\right)$ *и* $\|x - T_n\| \leqslant \dfrac{8k}{\log 2}\,\omega_2\!\left(\dfrac{1}{n}\right)$ *для* $n \geqslant 2$.

Теорема 2. *Если* $\omega(\delta)$ *удовлетворяет условиям* $(**)$ *и* $\|x - S_n\| \leqslant \omega(1/n)$, *то* $\omega_1(\delta) \leqslant 4M\omega(\delta)$ *для* $0 \leqslant \delta \leqslant 1$.

Теорема 3. *Система* $\{\varphi_n(t)\}$, $n = 0, 1, \ldots$, *является базисом Шаудера пространства* $C_{\langle 0,1\rangle}$ ([1], *стр.* 50) *и*

$$x(t) = \sum_{n=0}^{\infty} b_n \varphi_n(t).$$

З. ШИМАНСКИЙ, О ДЕФОРМИРУЕМОСТИ СЕРДЦЕВИНЫ ЯДРА . . . . стр. 233—236

В настоящей работе вычислена деформируемость сердцевины ядер с одним, внешним, по отношению к дважды замкнутой оболочке, нуклоном. Исходим из предпосылки, что ядерную форму определяет волновая функция, соответствующая потенциалу в форме прямоугольной ямы. Деформируемость для такого

потенциала превышает деформируемость для гармонического осциллатора, причем она уменьшается по мере роста числа нуклонов. Для тяжелых ядер она несколько больше измеряемой величины.

В. КРУЛИКОВСКИЙ, РАССЕИВАНИЕ МЕЗОНОВ K НА ЖЕСТКОМ ИСТОЧНИКЕ . . . . стр. 237—244

В работе дается трактовка типа Чю Лова проблемы рассеивания мезонов K жестким источником (барионом).

Опираясь на формализм сильных взаимодействий, формулировка которого дается автором настоящей работы, выводятся для рассеивания $N+K \to N+K$ в рассматриваемом приближении жесткого источника: 1) уравнение Лова, 2) дисперсионные формулы, а также дается в общих чертах 3) аппроксимация эффективного радиуса (*effective range approximation*), из которой вытекает возможность высоко энергетического резонанса $P_{3/2}$ в изотропных состояниях $T = 0, 1$.

В. КРУЛИКОВСКИЙ, ИЗОБАРЫ НУКЛЕОНА, ВЫЗВАННЫЕ ПОЛЕМ МЕЗОНОВ K В ТЕОРИИ ЖЕСТКОГО ИСТОЧНИКА . . . стр. 245—250

В настоящей работе рассматривается проблема проявления возбуждений нуклеона по отношению к полю мезонов K. Эта проблема аналогична проблеме существования изобаров нуклеона по отношению к полю мезонов π, дискутированной многими авторами при использовании метода сильного взаимодействия, а также автором настоящей работы при использовании некоторого метода, имеющего более общий характер. Примененный в настоящей работе формализм взаимодействия барионы-мезоны K был введен автором уже ранее.

Г. РАЙСКИЙ, ШЕСТИМЕРНАЯ ИНТЕРПРЕТАЦИЯ ЯДЕРНЫХ СИЛ . . . . стр. 251—254

Уравнение нуклона взаимодействующего с мезонным полем принимает особенно простой вид, если запишем его при использовании восьмикомпонентных спиноров. Это свидетельствует о том, что пространство Минковского является подпространством некоторого шестимерного пространства. Изотопный спин, электрический заряд, а также псевдоскалярный характер мезона π имеют простую интерпретацию в пределах шестимерной геометрии.

А. ЗАРЕМБА, ПРОВОДИМОСТЬ Ge, ВЫЗВАННАЯ ЭЛЕКТРОННОЙ БОМБАРДИРОВКОЙ . . . . стр. 255—260

Исследовалась проводимость Ge, связанная с ударной ионизацией, вызванной бомбардировкой монокристаллического образца Ge типа n электронами с энергией 22,7 кэв.

Целью настоящей работы является определение средней эффективной энергии ε, затрачиваемой на образование одной пары: электрон—дырка. Для определения этой величины, необходимо знать эффективное (связанное так с объемной как и с поверхностной рекомбинацией) время жизни τ генерируемых носителей тока. Так как электронная бомбардировка изменяет состояние поверхности и может повлиять на поверхностную рекомбинацию, в настоящей работе определено так ε как и τ, измеряя зависимость наведенной проводимости от приложенного на образец напряжения (явление эксклюзии).

Полученная кривая (рис. 3) позволяет найти ε и τ; для исследуемого образца получено: $\varepsilon = 2{,}8 \pm 0{,}3$ *эв*, $\tau = 1{,}8 \pm 0{,}2$. *μсек.*